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AN EXTENSION OF THE LAPLACE TRANSFORM TO SCHWARTZ DISTRIBUTIONS

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16. Abstract

A new characterization of the Laplace transform is developed which extends the transform to the Schwartz distributions. The class of distributions includes, in addition to all ordinary functions, the impulse functions and other singular functions which occur as solutions to ordinary and partial differential equations. The standard theorems on analyticity, uniqueness, and invertibility of the transform are proved by using the new characterization as the definition of the Laplace transform. The new definition uses sequences of linear transformations on the space of distributions in a manner suggested by a paper of E. Gesztelyi which extended the Laplace transform to another class of generalized functions, the Mikusiński operators. It is shown that the new sequential definition of the transform is equivalent to Schwartz' extension of the ordinary Laplace transform to distributions but, in contrast to Schwartz' definition, does not use the distributional Fourier transform.

Several theorems concerning the particular linear transformations used to define the Laplace transforms are proved. All the results proved in one dimension are extended to the n-dimensional case, but proofs are presented only for those situations that require methods different from their one-dimensional analogs.

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SUMMARY

A new characterization of the Laplace transform is developed which extends the transform to the Schwartz distributions. The class of distributions includes, in addition to all ordinary functions, the impulse functions and other singular functions which occur as solutions to ordinary and partial differential equations. The standard theorems on analyticity, uniqueness, and invertibility of the transform are proved by using the new characterization as the definition of the Laplace transform. The new definition uses sequences of linear transformations on the space of distributions in a manner suggested by a paper of E. Gesztelyi which extended the Laplace transform to another class of generalized functions, the Mikusiński operators. It is shown that the new sequential definition of the transform is equivalent to Schwartz' extension of the ordinary Laplace transform to distributions but, in contrast to Schwartz' definition, does not use the distributional Fourier transform.

Several theorems are proved concerning the application of exponential shifts and dilatations to distributions. In particular, the sequence formed by multiplying an integrable distribution and its independent variable by the sequence of positive integers converges as the integer index approaches infinity to a constant multiple of the delta distribution. The constant corresponds to the integral of the distribution. It is also proved that such a sequence can converge only if the original distribution is a distribution of slow growth. The limit of such a dilatation sequence must always be a linear combination of the delta distribution and the distribution corresponding to the Cauchy principal value of an improper integral. Moreover, such dilatation sequences and exponential shifts are used to define the Laplace transform of the original distribution.

All the results are extended to the n-dimensional case, but proofs are presented only for those situations that require methods different from their one-dimensional analogs.

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INTRODUCTION

The Laplace transform has been an important tool of applied mathematicians and engineers for many years. The properties of the ordinary Laplace transform have been well known at least since Widder published his book, "The Laplace Transform," (ref. 1) in 1941. L. Schwartz (ref. 2) extended the Laplace transform to distributions in 1952, and there have been many other extensions since then. (For example, see refs. 3 to 6.) In this report another characterization of the Laplace transform for distributions is given and is used to prove the standard theorems on analyticity, uniqueness, and invertibility of the transform.

The work which led to this study was motivated by a paper of E. Gesztelyi on linear operator transformations (ref. 7). Two classes of transformations he considers are the dilatations U_j and expontial shifts T^{-p} which are defined for ordinary functions f, complex numbers p, and positive integers j by

$$U_j f(t) = j f(jt)$$

$$T^{-p} f(t) = e^{-pt} f(t)$$

Gesztelyi shows that whenever the sequence $U_j f$ converges (in the sense of Mikusiński convergence (ref. 8)), the limit is necessarily a complex number. In addition, he proves that if f is a function which has a Laplace transform at p, then the sequence of functions $\left\{U_j T^{-p} f(t)\right\}$ converges (in the Mikusiński sense) as $j \to \infty$ to the classical Laplace transform of f at p. He then defines the Laplace transform of a Mikusiński operator x as the limit (whenever it exists in the sense of Mikusiński convergence) of the sequence $\left\{U_j T^{-p} x\right\}$, and shows that this definition generalizes the previous formulations of the Laplace transform for Mikusiński operators of G. Doetsch (ref. 9), and V. A. Ditkin and A. P. Prudnikov (refs. 10 and 11). Since the dilatations U_j and shifts T^{-p} may be defined on the space of Schwartz distributions, it was conjectured that there might be results analogous to Gesztelyi's results in this different setting.

It will be assumed that the reader of this report is familiar with the basic results concerning distributions and their test functions. These results can be found in any of the many readily available textbooks on the subject of distributions or generalized functions. In particular, the books of Zemanian (ref. 3) and Horváth (ref. 12) contain all the information required for a thorough understanding of this report.

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Denote by $\mathcal{D}(R^n)$ the space of all infinitely differentiable complex-valued functions of the n-dimensional real variable $t = t_1, t_2, \ldots, t_n$ with compact support. If $j = j_1, j_2, \ldots, j_n$ is a multi-index, then $\phi^{(j)}(t)$ or $\partial^j \phi(t)$ denotes

$$\frac{\partial^{|j|}\phi\left(t_1, \ldots, t_n\right)}{\partial t_1^{j_1}\, \partial t_2^{j_2} \ldots \partial t_n^{j_n}}$$

where $|j| = j_1 + j_2 + \ldots + j_n$. A sequence $\left\{\phi_k\right\}$ in $\mathcal{D}(R^n)$, converges to zero in $\mathcal{D}(R^n)$ as $k \to \infty$ if there is a fixed compact set C containing the support of every ϕ_k; and for every multi-index j, $\left\{\phi_k^{(j)}\right\}$ converges to zero uniformly on C as $k \to \infty$.

Denote by $\mathcal{D}'(R^n)$ the space of all linear transformations f from $\mathcal{D}(R^n)$ to the complex field which are continuous in the sense that if $\left\{\phi_k\right\}$ converges to zero in $\mathcal{D}(R^n)$, then the sequence of complex numbers $\left\{\left\langle f, \phi_k\right\rangle\right\}$ converges to zero as $k \to \infty$. Although there are several different ways to assign topologies to $\mathcal{D}(R^n)$ and determine the set $\mathcal{D}'(R^n)$ of continuous linear functionals on $\mathcal{D}(R^n)$, no topology will be defined explicitly here since the notion of sequential continuity is sufficient for the needs of this report. The elements of $\mathcal{D}'(R^n)$ are the distributions defined by L. Schwartz in reference 2. In the sequel, when the dimension of the space R^n is understood, $\mathcal{D}$ and $\mathcal{D}'$ will be written for $\mathcal{D}(R^n)$ and $\mathcal{D}'(R^n)$, respectively.

Let $\mathcal{S}(R^n)$ (or $\mathcal{S}$) denote the space of infinitely differentiable complex-valued functions of $t = t_1, t_2, \ldots, t_n$ which approach zero faster than any power of $1/|t|$ as $|t| \to \infty$. Give $\mathcal{S}$ the locally convex topology defined by the family $\left\{q_{k,j}\right\}$ of seminorms where

$$q_{k,j}(\phi) = \max\left\{\left|(1 + t^2)^k \partial^j\ \phi(t)\right| : t \in R^n\right\}$$

for every positive integer k and multi-index j. The space $\mathcal{S}'$ of weakly continuous linear functionals on $\mathcal{S}$ consists of the tempered distributions or distributions of slow growth.

Let $\mathcal{E}(R^n)$ (or $\mathcal{E}$) denote the space of all infinitely differentiable complex-valued functions on R^n. For each compact set C and each multi-index j, define the seminorm $q_{C,j}$ by

$$q_{C,j}(\phi) = \max\left\{\left|\partial^j \phi(t)\right| : t \in C\right\}$$

Equip $\mathcal{E}$ with the locally convex topology defined by the family $\left\{q_{C,j}\right\}$ of seminorms, and let $\mathcal{E}'(R^n)$ or $\mathcal{E}'$ denote the space of weakly continuous linear functionals on $\mathcal{E}$. Then $\mathcal{E}'$ is the space of distributions of compact support. It follows from standard results in the theory of distributions that $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$, that $\mathcal{D}$ is dense in both $\mathcal{S}$ and $\mathcal{E}$ with their respective topologies, and that $\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$.

In the next section the space $\mathcal{B}$ of bounded infinitely differentiable functions and its subset $\mathcal{B}_0$ consisting of those functions in $\mathcal{B}$ which converge to zero along with each derivative as $|t| \to \infty$ are introduced. Distributions in $\mathcal{B}_0'$, sometimes called integrable distributions, are characterized as those which satisfy certain a priori bounds when applied to test functions in $\mathcal{D}$, and it is shown that each distribution in $\mathcal{B}_0'$ may be extended to all of $\mathcal{B}$.

In the third section the linear transformations U_j and T^{-p} are introduced. It is shown that if f is in $\mathcal{B}_0'$, then $\left\{U_j f\right\}$ converges in $\mathcal{D}'$ as $j \to \infty$ to $\langle f, 1\rangle \delta$. Also, distributions h that are limits of sequences of the form $\left\{U_j f\right\}$ are characterized as linear combinations of $\delta(t)$ and p.v. $\frac{1}{t}$. This characterization gives an example $\left(\text{p.v. } \frac{1}{t}\right)$ of a distribution f which is not in $\mathcal{B}_0'$, but for which the sequence $\left\{U_j f\right\}$ converges in $\mathcal{D}'$. The distribution p.v. $\frac{1}{t}$ is in $\mathcal{S}'$, however, and it is proved that the sequence $\left\{U_j f\right\}$ can converge in $\mathcal{D}'$ only if f is in $\mathcal{S}'$.

In the fourth section the Laplace transform of a distribution f is defined by

$$L[f](p) = \frac{1}{\phi(0)} \lim_{j\to\infty} \left\langle U_j T^{-p} f, \phi\right\rangle \tag{1}$$

where $\phi(0) \neq 0$. Theorem 3 shows that if $T^{-p_1}f$ and $T^{-p_2}f$ are both in $\mathcal{S}'$, then definition (1) may be used for all complex numbers p with $\mathrm{Re}\, p_1 < \mathrm{Re}\, p < \mathrm{Re}\, p_2$. This definition is used to prove the standard properties of analyticity, invertibility, and uniqueness of the Laplace transform.

Since Schwartz was the first to extend the Laplace transform to distributions, all other extensions (including those of Zemanian and Ishihara) are compared with his in the references. It is shown in the fourth section that definition (1) is equivalent to Schwartz's definition of the transform.

In the fifth section the results of the third and fourth sections are extended to distributions in $\mathcal{S}'(R^n)$. The extensions are, for the most part, straightforward; so only those which require basically new methods in n-dimensions are proved. The appendix contains the construction of a partition of unity used several places in the text and the proofs of several lemmas needed in the text.

SYMBOLS AND NOTATION

Due to the theoretical nature of this report, most of the symbols are used in a generic rather than in a specific sense. For this reason, the symbol list will be divided into three parts. First, the symbols used generically will be listed in groups according to their usage. Next, the symbols with specific meaning will be listed alphabetically. Finally, the mathematical symbols used in the report will be listed.

Generic Symbols

$r,t,\xi,\sigma,\tau,\omega$	real independent variables, dimension may be $\geqq 1$
p,q	complex independent variables
$\gamma,\theta,\lambda,\rho,\phi,\psi$	test functions
f,g,h	distributions
F	analytic functions of the complex variable p
i,j,k,m,ℓ,ν	indices (nonnegative integers)
j,**k**	multi-indices (n-tuples of nonnegative integers). It is specified in the text whether **j** and **k** have dimension 1 or n
I,J	index sets for sums or unions
M,N,n	positive integer constants

η, μ	complex constants
c, d, α, β	real constants
a,b	n-dimensional real constants
C	compact subsets of R^n
ϵ	small positive constants
K,L,P	positive numbers used as bounds
Ω	domains of definition for Laplace transform
x	Mikusiński operators

Any of the generic symbols can have subscript indices or superscript primes to indicate different elements of the same type.

Specific Symbols

Spaces:

$\mathscr{B}(R^n)$	space of infinitely differentiable complex-valued functions of an n-dimensional real variable, each of whose derivatives is bounded
$\mathscr{B}'(R^n)$	space of continuous linear functionals from $\mathscr{B}(R^n)$ to the space of complex numbers – the dual space of $\mathscr{B}(R^n)$
$\mathscr{B}_0(R^n)$	space of infinitely differentiable complex-valued functions of an n-dimensional real variable, each of whose derivatives approaches zero at infinity
$\mathscr{B}_0'(R^n)$	space of continuous linear functionals from $\mathscr{B}_0(R^n)$ to the space of complex numbers – the dual space of $\mathscr{B}_0(R^n)$ – sometimes called integrable distributions
$\mathscr{D}(R^n)$	space of infinitely differentiable complex-valued functions of an n-dimensional real variable, each of which is zero except in a compact subset of R^n

$\mathcal{D}'(R^n)$ space of continuous linear functionals from $\mathcal{D}(R^n)$ to the space of complex numbers – the dual space of $\mathcal{D}(R^n)$ – the Schwartz distributions

C^n space of n-tuples of complex numbers

$\mathcal{E}(R^n)$ space of infinitely differentiable complex-valued functions of an n-dimensional real variable

$\mathcal{E}'(R^n)$ space of continuous linear functionals from $\mathcal{E}(R^n)$ to the space of complex numbers – the dual space of $\mathcal{E}(R^n)$ – the compact distributions

L^1 space of integrable functions of a real variable

R^n space of n-tuples of real numbers; when $n = 1$, this space is denoted R

$\mathcal{S}(R^n)$ space of infinitely differentiable functions of an n-dimensional real variable, each of whose derivatives approaches zero faster than every power of the independent variable – the functions of rapid descent

$\mathcal{S}'(R^n)$ space of continuous linear functionals from $\mathcal{S}(R^n)$ to the space of complex numbers – the dual space of $\mathcal{S}(R^n)$ – the tempered distributions or distributions of slow growth

Other symbols:

$$A = \int_{-1/2}^{1/2} \exp\left[\frac{1}{4t^2 - 1}\right] dt$$

$\mathcal{F}$ Fourier transform (sometimes denoted by $(\tilde{\ })$)

$\mathcal{F}^{-1}$ inverse Fourier transform

$G(\omega)$ function defined in the proof of theorem 12

Im imaginary part of a complex number

L Laplace transform

p.v. $\frac{1}{t}$	distribution defined in example 2
q_j	seminorm used to define topology of space $\mathcal{B}(R^n)$
$q_{k,j}$	seminorm used to define topology of space $\mathcal{J}(R^n)$
$q_{C,j}$	seminorm used to define topology of space $\mathcal{E}(R^n)$
Re	real part of a complex number
T^{-p}	a linear transformation on the space of distributions – called exponential shift
U_a	a linear transformation on the space of distributions – called dilatation transformation
$\{\gamma_k\}$	a partition of unity defined in the appendix
Γ_r	path for integration of analytic function used in proof of theorem 12
δ	delta distribution defined for every test function ϕ by $\langle \delta, \phi \rangle = \phi(0)$

Mathematical Symbols

$\langle f, \phi \rangle$	evaluation of a distribution f at test function ϕ – a complex number. (If f is a locally integrable function, then $\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\ \phi(t)\ dt.$)
$\otimes$	direct product or tensor product
$*$	convolution
$\sum$	summation
$\{A : B\}$	set of all A such that B is true
$\cup$	union
$\subset$	containment – means "is a subset of"

$\in$	containment – means "is an element of"
$\lvert\ \rvert$	Euclidean norm in R^n (or the order of a multi-index j, $\lvert j \rvert = j_1 + j_2 + \cdots + j_n$)
max	maximum
sup	supremum
1	in addition to the positive integer one, this sometimes represents the function whose value is always the positive integer one
$\check{f}$	distribution defined for every test function ϕ by $\langle \check{f}(t), \phi(t) \rangle = \langle f(t), \phi(-t) \rangle$; similarly for g
$\hat{f}$	an extension of distribution f – defined in theorem 2
$\tilde{\phi}$	Fourier transform of ϕ

THE SPACE $\mathcal{B}_0'$

Denote by $\mathcal{B}(R^n)$ (or, where R^n is understood, by $\mathcal{B}$) the space of all complex-valued functions of an n-dimensional real variable $t = t_1, t_2, \ldots, t_n$ which possess continuous and bounded partial derivatives of all orders. For each multi-index j, define the seminorm q_j on $\mathcal{B}$ by

$$q_j(\phi) = \sup\left\{ \left| \partial^j \phi(t) \right| : t \in R^n \right\} \tag{2}$$

and equip $\mathcal{B}$ with the locally convex topology determined by the family of seminorms $\{q_j\}$. (For convenience, hereafter, $\sup f(t)$ or $\sup\limits_t f(t)$ will denote $\sup\{f(t) : t \in R^n\}$.)

A sequence $\{\phi_k\}$ converges in $\mathcal{B}$ to a function ϕ with respect to this topology if, and only if, each derived sequence $\{\partial^j \phi_k\}$ converges uniformly to $\partial^j \phi$.

It is easy to see that $\mathcal{D} \subset \mathcal{B} \subset \mathcal{E}$. The subspace $\mathcal{D}$ is not dense in $\mathcal{B}$, however, because the constant function $1(t)$ is in $\mathcal{B}$ but cannot be uniformly approximated by functions in $\mathcal{D}$, since for any ϕ in $\mathcal{D}$

$$q_0\left[\phi(t) - 1(t)\right] = \sup\left|\phi(t) - 1(t)\right| \geqq 1$$

For this reason the dual space $\mathcal{B}'$ of $\mathcal{B}$ cannot be identified with a subspace of the space $\mathcal{D}'$ of distributions. In fact Zemanian (ref. 4) demonstrates this condition by giving an example of a nonzero generalized function in $\mathcal{B}'$ whose restriction to $\mathcal{D}$ is the zero distribution.

Since this work will be confined to the class $\mathcal{D}'$ of distributions, it is necessary to consider a subspace $\mathcal{B}_0$ of $\mathcal{B}$ consisting of those functions in $\mathcal{B}$ each of whose derivatives approach zero as $|t| \to \infty$. Specifically, a function ϕ is in $\mathcal{B}_0$ if, and only if, ϕ is in $\mathcal{B}$ and for each multi-index j and each positive number ϵ, there is a compact set $C_{j\epsilon}$ such that if t is not in $C_{j\epsilon}$, then $\left|\partial^j \phi(t)\right| < \epsilon$.

Give $\mathcal{B}_0$ the topology induced by $\mathcal{B}$ which makes $\mathcal{B}_0$ a locally convex topological vector space. To see that $\mathcal{D}$ is dense in $\mathcal{B}_0$, let $\left\{\theta_k\right\}$ be a sequence of functions in $\mathcal{D}$ such that

$$\theta_k(t) = 1 \qquad \left(|t| \leqq k\right)$$

$$\theta_k(t) = 0 \qquad \left(|t| > k + 1\right)$$

$$\sup\left|\theta_k^{(j)}(t)\right| \leqq \sup\left|\theta_1^{(j)}(t)\right|$$

for every multi-index j. If ϕ is a function in $\mathcal{B}_0$, then $\left\{\theta_k\phi\right\}$ is a sequence in $\mathcal{D}$ that converges in $\mathcal{B}_0$ to ϕ, which shows that $\mathcal{D}$ is dense in $\mathcal{B}_0$. Therefore, the dual space $\mathcal{B}_0'$ of $\mathcal{B}_0$ is a subspace of $\mathcal{D}'$ and a distribution f in $\mathcal{B}_0'$ is completely determined by its values on $\mathcal{D}$. The following theorem is a useful characterization of distributions in $\mathcal{B}_0'$.

Theorem 1: A distribution f is in $\mathscr{B}_0'$ if, and only if, there is a number K such that

$$\left|\langle f, \phi\rangle\right| \leqq K \max_{|j| \leqq K} \sup_t \left|\phi^{(j)}\right| \tag{3}$$

for every ϕ in $\mathscr{D}$.

Proof: To prove that condition 3 implies f belongs to $\mathscr{B}_0'$, let $\left\{\phi_k\right\}$ be a sequence in $\mathscr{D}$ that converges to zero in the topology of $\mathscr{B}_0$. Then $\sup\left|\phi_k^{(j)}\right| \to 0$ as $k \to \infty$ for every j and

$$\lim_{k\to\infty}\left|\left\langle f, \phi_k\right\rangle\right| \leqq \lim_{k\to\infty} K \max_{|j| \leqq K} \sup_t \left|\phi_k^{(j)}\right| = 0$$

To show that f can be defined on all $\mathscr{B}_0$, let ϕ be a function in $\mathscr{B}_0$ and $\left\{\phi_k\right\}$ a sequence in $\mathscr{D}$ that converges in $\mathscr{B}_0$ to ϕ. Then the set $\left\{K \max_{|j| \leqq K} \sup_t \left|\phi_k^{(j)}\right|\right\}$ is bounded above and so $\left\{\left|\left\langle f, \phi_k\right\rangle\right|\right\}$ is also bounded above. Since $\left\{\phi_k - \phi_\ell\right\}$ is a sequence in $\mathscr{D}$ that converges to zero in $\mathscr{B}_0$ as k and ℓ tend to infinity independently,

$$\lim_{k,\ell\to\infty}\left\langle f, \phi_k - \phi_\ell\right\rangle = 0$$

and $\left\{\left\langle f, \phi_k\right\rangle\right\}$ is a Cauchy sequence with a finite limit.

Define $\langle f, \phi\rangle = \lim_{k\to\infty}\left\langle f, \phi_k\right\rangle$. If $\left\{\psi_k\right\}$ is another sequence in $\mathscr{D}$ that converges in $\mathscr{B}_0$ to ϕ, then $\left\{\phi_k - \psi_k\right\}$ converges in $\mathscr{B}_0$ to zero, so $\langle f, \phi\rangle$ is well defined. Since $\mathscr{D}$ is dense in $\mathscr{B}_0$, f is extended to all of $\mathscr{B}_0$ and the extension is clearly linear. To see that f is continuous on $\mathcal{B}_0$, notice that $\left|\langle f, \phi\rangle\right| \leqq K \max_{|j|\leqq K} \sup\left|\phi^{(j)}\right|$ holds even for $\phi \in \mathcal{B}_0$. Thus f is in $\mathcal{B}_0'$.

The proof that condition 3 holds if f is in $\mathcal{B}_0'$ proceeds by contradiction. Suppose that $\left\{\phi_k\right\}$ is a sequence in $\mathcal{B}_0$ such that for each k,

$$\left|\left\langle f, \phi_k\right\rangle\right| > k \max_{|j|\leqq k} \sup_t \left|\phi_k^{(j)}(t)\right| = k \max_{|j|\leqq k} q_j\left(\phi_k\right)$$

and define $\theta_k = \dfrac{\phi_k}{k \max\limits_{|j|\leqq k} q_j\left(\phi_k\right)}$. Then θ_k is in $\mathcal{B}_0$ for every k and

$$q_m\left(\theta_k\right) = \frac{q_m\left(\phi_k\right)}{k \max\limits_{|j|\leqq k} q_j\left(\phi_k\right)} \leqq \frac{1}{k} \qquad (k \geqq m)$$

so $\theta_k \to 0$ in $\mathcal{B}_0$ as $k \to \infty$. Since f is in $\mathcal{B}_0'$, this statement means that $\left\langle f, \theta_k\right\rangle \to 0$. However, by the definition of θ_k,

$$\left|\left\langle f, \theta_k\right\rangle\right| = \frac{\left|\left\langle f, \phi_k\right\rangle\right|}{k \max\limits_{|j|\leqq k} q_j\left(\phi_k\right)} > 1$$

This statement contradicts the fact that $\left\langle f, \theta_k\right\rangle \to 0$, and thus there can be no such sequence $\left\{\phi_k\right\}$. Therefore, if f is in $\mathcal{B}_0'$, condition 3 holds, and the proof is complete.

Since $\mathcal{B}_0 \subset \mathcal{B}$ and the topology of $\mathcal{B}_0$ is that induced by $\mathcal{B}$, each element of $\mathcal{B}'$ has a restriction to $\mathcal{B}_0$, that is, in $\mathcal{B}_0'$. The next theorem shows that a converse is also true, that is, that each element of $\mathcal{B}_0'$ can be extended to all $\mathcal{B}$.

<u>Theorem 2</u>: Each distribution f in $\mathcal{B}_0'$ has a unique extension $\hat{f}$ in $\mathcal{B}'$ with the property that $\left\langle \hat{f}, \phi_k\right\rangle$ converges to $\left\langle \hat{f}, \phi\right\rangle$

whenever $\left\{\phi_k\right\}$ is a uniformly bounded sequence in $\mathcal{B}$ that converges to ϕ with respect to the topology induced on $\mathcal{B}$ by $\mathcal{E}$.

Proof: If f is in $\mathcal{B}_0'$, then by theorem 1 there is a number K such that for every ψ in $\mathcal{B}_0$,

$$\left|\left\langle f, \psi\right\rangle\right| \leqq K \max_{|j| \leqq K} \sup_t \left|\partial^j \psi(t)\right|$$

Let ϕ be in $\mathcal{B}$ and suppose I is a finite set of nonnegative integers. Let $\left\{\gamma_i\right\}_{i=1}^{\infty}$ be the partition of unity defined in the appendix. Then

$$\left|\left\langle f, \sum_{i \in I} \gamma_i \phi\right\rangle\right| \leqq K \max_{|j| \leqq K} \sup_t \left|\partial^j\left(\sum_{i \in I} \gamma_i \phi\right)(t)\right| \leqq K \max_{|j| \leqq K} \sup_t \left|\sum_{k \leqq j} \binom{j}{k} \left(\sum_{i \in I} \gamma_i\right)^{(k)} \phi^{(j-k)}(t)\right|$$

$$\leqq KK^n (K!)^n \max_{|k| \leqq K} \sup_t \left|\gamma_0^{(k)}(t)\right| \max_{|j| \leqq K} \sup_t \left|\phi^{(j)}(t)\right| = P$$

Since P is independent of the choice of the set I, lemma 1 given in the appendix implies that for any finite set I of nonnegative integers,

$$\sum_{i \in I} \left|\left\langle f, \gamma_i \phi\right\rangle\right| \leqq 4P$$

Therefore

$$\sum_{i=0}^{\infty} \left|\left\langle f, \gamma_i \phi\right\rangle\right| \leqq 4P$$

and the series $\sum_{i=0}^{\infty} \left\langle f, \gamma_i \phi\right\rangle$ converges absolutely.

Define an extension $\hat{f}$ of f by

$$\left\langle \hat{f}, \phi \right\rangle = \sum_{i=0}^{\infty} \left\langle f, \gamma_i \phi \right\rangle .$$

for every ϕ in B. To see that $\hat{f} = f$ on $\mathcal{B}_0$, notice that if $\psi \in \mathcal{B}_0$, then $\sum_{i=0}^{\infty} \left\langle f, \gamma_i \psi \right\rangle$ is also absolutely convergent, so

$$\left\langle \hat{f}, \psi \right\rangle = \sum_{i=0}^{\infty} \left\langle f, \gamma_i \psi \right\rangle = \left\langle f, \sum_{i=0}^{\infty} \gamma_i \psi \right\rangle = \left\langle f, \psi \right\rangle$$

It remains to be shown that $\hat{f}$ is continuous on $\mathcal{B}$. This continuity will follow from the second part of the proof which shows that $\hat{f}$ is continuous even with respect to weaker topology than the one given in $\mathcal{B}$. To this end, let $\left\{ \phi_k \right\}$ be a uniformly bounded sequence in $\mathcal{B}$ that converges to zero in the topology of $\mathcal{E}$, and let

$$P_K = \max_{|j| \leq K} \sup_k \sup_t \left| \partial^j \phi_k(t) \right|$$

where K is the constant defined for f by theorem 1. Let I be a finite set of nonnegative integers and for each $i \in I$, let k_i be a positive integer. Then

$$\left| \left\langle f, \sum_{i \in I} \gamma_i \phi_{k_i} \right\rangle \right| \leq K \max_{|j| \leq K} \sup_t \left| \partial^j \left(\sum_{i \in I} \gamma_i \phi_{k_i} \right) \right| \leq KK^n (K!)^n P_K \max_{|j| \leq K} \sup_t \left| \partial^j \gamma_0(t) \right| = P'$$

Therefore, for every finite subset I of nonnegative integers and every choice of the collection $\left\{ k_i \right\}$ of positive integers,

$$\sum_{i \in I} \left| \left\langle f, \gamma_i \phi_{k_i} \right\rangle \right| \leq 4P' \tag{4}$$

It is already known that for each k, $\sum_{i=0}^{\infty} \left\langle f, \gamma_i \phi_k \right\rangle$ converges absolutely. It will be shown that this convergence is uniform with respect to k. Let ϵ be a positive number. Then for each k the absolute convergence of $\sum_{i=0}^{\infty} \left\langle f, \gamma_i \phi_k \right\rangle$ guarantees the existence of a smallest positive integer N_k such that

$$\sum_{i=N_k+1}^{\infty} \left| \left\langle f, \gamma_i \phi_k \right\rangle \right| < \frac{\epsilon}{2} \tag{5}$$

Suppose that the set $\left\{N_k\right\}$ cannot be bounded above. $\Big($Assume $N_k > 1$ for every k, choosing, if necessary, a subsequence of $\left\{\phi_k\right\}$ for which this is true.$\Big)$

Since N_k is the smallest positive integer that satisfies relation (5), there must also be positive integers $\left\{M_k\right\}$ such that for each k

$$\sum_{i=N_k}^{M_k} \left| \left\langle f, \gamma_i \phi_k \right\rangle \right| \geqq \frac{\epsilon}{2} \tag{6}$$

Pick a sequence of positive integers $\left\{\nu_k\right\}$ in the following way. Let $\nu_1 = N_1$. Since $M_1 < \infty$, there is an integer ν_2 such that $N_{\nu_2} > M_1$. Similarly, for each k, pick ν_k such that $N_{\nu_k} > M_{\nu_{k-1}}$. Then if M is a positive integer which is larger than $8P'/\epsilon$, inequality (6) guarantees that

$$\sum_{k=1}^{M} \sum_{i=N_{\nu_k}}^{M_{\nu_k}} \left| \left\langle f, \gamma_i \phi_{\nu_k} \right\rangle \right| \geqq M \frac{\epsilon}{2} > 4P'$$

But this expression is a sum of the form

$$\sum_{i \in I} \left| \left\langle f, \gamma_i \phi_{k_i} \right\rangle \right|$$

given in expression (4), where the finite set

$$I = \bigcup_{1 \leqq k \leqq M} \ \bigcup_{N_{\nu_k} \leqq i \leqq M_{\nu_k}} \left\{ i \right\}$$

and $k_i = \nu_k$ for $N_{\nu_k} \leqq i \leqq M_{\nu_k}$. The assumption that the set $\left\{ N_k \right\}$ is unbounded has led to a contradiction. Therefore it may be assumed that there is a positive integer N such such that $N_k \leqq N$ for every k, and so for every k,

$$\sum_{i=N+1}^{\infty} \left| \left\langle f, \gamma_i \phi_k \right\rangle \right| < \frac{\epsilon}{2} \tag{7}$$

Now $\left\{ \phi_k \right\}$ converges to zero in the topology of $\mathcal{E}$, and the derivatives of γ_i are uniformly bounded for all i; thus there must be a positive integer N' such that if $k > N'$, then

$$K \max_{|j| \leqq K} \sup_t \left| \partial^j \left(\sum_{i=0}^{N} \gamma_i \phi_k \right) \right| < \frac{\epsilon}{2} \tag{8}$$

Then, by expressions (7) and (8), as long as $k \geqq N'$

$$\left| \left\langle \hat{f}, \phi_k \right\rangle \right| = \left| \sum_{i=0}^{\infty} \left\langle f, \gamma_i \phi_k \right\rangle \right| \leqq \left| \left\langle f, \sum_{i=0}^{N} \gamma_i \phi_k \right\rangle \right| + \sum_{i=N+1}^{\infty} \left| \left\langle f, \gamma_i \phi_k \right\rangle \right|$$

$$\leqq K \max_{|j| \leqq K} \sup_t \left| \partial^j \left(\sum_{i=0}^{N} \gamma_i \phi_k \right) \right| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $\lim_{k \to \infty} \left\langle \hat{f}, \phi_k \right\rangle = 0$. This statement proves that $\hat{f}$ is sequentially continuous. Since $\mathcal{B}$ is Hausdorff and the topology of $\mathcal{B}$ is defined by a countable family of seminorms, $\mathcal{B}$ is metrizable. Thus, sequential continuity of $\hat{f}$ on $\mathcal{B}$ guarantees that $\hat{f}$ is in $\mathcal{B}'$.

The remarks at the beginning of this section show that there may be more than one way to extend a distribution f in $\mathcal{B}_0'$ to all of $\mathcal{B}$. However, $\mathcal{D}$ is dense in $\mathcal{E}$ and, therefore, $\mathcal{D}$ is dense in $\mathcal{B}$ with the topology induced by $\mathcal{E}$. Moreover, if ϕ is in $\mathcal{B}$, there is a uniformly bounded sequence $\left\{\phi_k\right\}$ in $\mathcal{D}$ that converges to ϕ in this topology. Thus, any two extensions of f which satisfy the property of the theorem must also be equal on $\mathcal{B}$. That is, there can be only one such extension, and $\hat{f}$ is unique. The proof of theorem 2 is now complete.

In the sequel, whenever a distribution f in $\mathcal{B}_0'$ is applied to a test function ϕ in $\mathcal{B}$, this will be understood to mean $\left\langle \hat{f}, \phi \right\rangle$, where $\hat{f}$ is the particular (unique) extension of f defined in theorem 2. In particular, the constant functions are in $\mathcal{B}$, so if f is in $\mathcal{B}_0'$, $\left\langle f, c \right\rangle = c\left\langle f, 1 \right\rangle$ is defined. If f happens to be a regular distribution in $\mathcal{B}_0'$ determined by an integrable function f(t), then

$$\left\langle f, 1 \right\rangle = \int_{R^n} f(t)\, dt$$

For this reason, distributions in $\mathcal{B}_0'$ are frequently called integrable distributions.

There are two more results concerning $\mathcal{B}_0'$ which will be needed in later sections. Recall that if a and b are in R^n, $a < b$ means $a_i < b_i$ $(i = 1, 2, \ldots, m)$ and e^{at} is the function $\exp\left[\sum_{i=1}^{n} a_i t_i\right]$.

<u>Theorem 3</u>: If f is in $\mathcal{D}'(R^n)$ and a, b are in R^n with $a < b$ such that $e^{-at} f(t)$ and $e^{-bt} f(t)$ are both in $\mathcal{S}'$, then for every n-dimensional complex number p with $a < \operatorname{Re} p < b$, $e^{-pt} f(t)$ is in $\mathcal{B}_0'$.

Proof: Let p be an n-dimensional complex number with $a < \operatorname{Re} p < b$, and let ϵ be a positive number in R^n such that

$$\epsilon_i < \min\left\{\text{Re } p_i - a_i,\ b_i - \text{Re } p_i:\ i = 1, 2, \ldots, n\right\}$$

If $\lambda(t) = e^{\epsilon t} + e^{-\epsilon t}$, then $\lambda(t)\, e^{-pt} f(t)$ is in $\mathcal{S}'$ and $1/\lambda(t)$ is in $\mathcal{S}$. Also, for every $\phi \in \mathcal{B}_0, \frac{\phi}{\lambda}(t)$ is in $\mathcal{S}$, so one may write

$$\left\langle e^{-pt} f(t),\ \phi(t)\right\rangle = \left\langle \lambda(t)\, e^{-pt} f(t),\ \frac{\phi}{\lambda}(t)\right\rangle$$

This expression clearly identifies $e^{-pt} f(t)$ as a continuous linear transformation on $\mathcal{B}_0$ as long as $a < \text{Re } p < b$, so the theorem is proved.

> Theorem 4: If f and g are in $\mathcal{B}_0'(R^n)$, then their convolution can be defined and is also in $\mathcal{B}_0'(R^n)$.

Proof: If f is in $\mathcal{D}'$, then let $\check{f}$ denote the distribution defined for every $\phi \in \mathcal{D}$ by

$$\left\langle \check{f},\ \phi\right\rangle = \left\langle f(t),\ \phi(-t)\right\rangle$$

Using the tensor product $\otimes$ to formally define $f * g$, gives

$$\left\langle f * g,\ \phi\right\rangle = \left\langle f(t) \otimes g(\tau),\ \phi(t + \tau)\right\rangle = \left\langle f(t),\ \left\langle g(\tau),\ \phi(t + \tau)\right\rangle\right\rangle$$
$$= \left\langle f(t),\ \left\langle \check{g}(\tau),\ \phi(t - \tau)\right\rangle\right\rangle = \left\langle f(t), \left(\check{g} * \phi\right)(t)\right\rangle$$

This string of equalities will be justified and the convolution $f * g$ will be defined as a distribution in $\mathcal{B}_0'$ if it can be shown that $\check{g} * \phi$ is in $\mathcal{B}$ when ϕ is in $\mathcal{B}_0$ and g is in $\mathcal{B}_0'$, and that $\check{g} * \phi_k$ converges to zero in $\mathcal{B}$ whenever ϕ_k converges to zero in $\mathcal{B}_0$. To do this, consider

$$\sup_t \left|\partial^j(\check{g} * \phi)(t)\right| = \sup_t \left|\partial^j\left\langle \check{g}(\tau),\ \phi(t - \tau)\right\rangle\right| = \sup_t \left|\partial^j\left\langle g(\tau),\ \phi(t + \tau)\right\rangle\right| = \sup_t \left|\left\langle g(\tau),\ \phi^{(j)}(t + \tau)\right\rangle\right|$$
$$\leq \sup_t K \max_{|i| \leq K} \sup_\tau \left|\partial^i\ \phi^{(j)}(t + \tau)\right| = K \max_{|i| \leq K} \sup_t \left|\phi^{(i+j)}(t)\right| = P_{K,j}$$

where K is the constant defined for g by theorem 1.

Therefore $\check{g} * \phi$ is in $\mathcal{B}$; and if ϕ_k converges to zero in $\mathcal{B}_0$, then $\sup_t \left|\phi_k^{(i+j)}(t)\right|$ converges to zero, and $\check{g} * \phi_k$ must converge to zero in $\mathcal{B}$. Thus, $\left\langle f * g, \phi \right\rangle = \left\langle f, \check{g} * \phi \right\rangle$ defines $f * g$ as a distribution in $\mathcal{B}_0'$, and the theorem is proved.

In the sequel the fact that $\mathcal{B}_0'$ is a subset of $\mathcal{S}'$ will frequently be used. This is easily seen to be true, since $\mathcal{S} \subset \mathcal{B}_0$ and $\mathcal{S}$ is dense in $\mathcal{B}_0$ with respect to the topology of $\mathcal{B}_0$. Another way of verifying that $\mathcal{B}_0' \subset \mathcal{S}'$ is to compare theorem 1 with the corresponding result for $\mathcal{S}'$ (Zemanian, ref. 3, p. 111).

THE TRANSFORMATIONS U_j AND T^{-p}

This section contains definitions and some results concerning two linear transformations on the space $\mathcal{D}'(R)$. The generalizations to $\mathcal{D}'(R^n)$ of these results will be postponed to a later section.

If $a > 0$ in R, define the linear transformation U_a on $\mathcal{D}'(R)$ by

$$\left\langle U_a\, f(t),\ \phi(t)\right\rangle = \left\langle a\, f(at),\ \phi(t)\right\rangle = \left\langle f(t),\ \phi\left(\frac{t}{a}\right)\right\rangle \tag{9}$$

for every distribution f and every test function ϕ. It can easily be verified that U_a is continuous and linear on $\mathcal{D}'(R)$.

Another useful transformation on $\mathcal{D}'(R)$ is defined in the following way: For each complex number p, let T^{-p} be defined by

$$\left\langle T^{-p} f(t),\ \phi(t)\right\rangle = \left\langle e^{-pt} f(t),\ \phi(t)\right\rangle \tag{10}$$

for each distribution f and test function ϕ. The transformation T^{-p} is clearly continuous and linear on $\mathcal{D}'(R)$.

The primary concern of this section is the convergence of the sequences of distributions $\left\{U_j f\right\}$ or $\left\{U_j T^{-p} f\right\}$ as $j \to \infty$. The first theorem is a direct corollary to theorem 2.

<u>Theorem 5</u>: If f is in $\mathcal{B}_0'$, then

$$\lim_{j\to\infty} U_j f = \left\langle f, 1\right\rangle \delta$$

where the limit is taken in $\mathcal{D}'$.

Proof: Let ϕ be in $\mathcal{D}$ and for each positive integer j let $\phi_j(t) = \phi\left(\frac{t}{j}\right)$. Then ϕ_j is also in $\mathcal{D}$ for each j and the sequence ϕ_j converges uniformly on compact sets as $j \to \infty$ to the function $\phi(0)1$. Also, if $k \geqq 1$, the sequence

$$\left\{\phi_j^{(k)}(t)\right\} = \left\{\left(\frac{1}{j}\right)^k \phi^{(k)}\left(\frac{t}{j}\right)\right\}$$

converges uniformly on compact sets to zero as $j \to \infty$. Therefore, the uniformly bounded sequence $\left\{\phi_j\right\}$ in $\mathcal{B}$ converges with respect to the topology induced on $\mathcal{B}$ by $\mathcal{E}$, and by theorem 2,

$$\lim_{j\to\infty} \left\langle U_j f, \phi\right\rangle = \lim_{j\to\infty} \left\langle f, \phi_j\right\rangle = \left\langle f, \phi(0)1\right\rangle = \left\langle\left\langle f, 1\right\rangle \delta, \phi\right\rangle$$

Thus

$$\lim_{j\to\infty} U_j f = \left\langle f, 1\right\rangle \delta$$

and the proof is complete.

An obvious question to ask is: Does the sequence $\left\{U_j f\right\}$ ever converge if f is not in $\mathcal{B}_0'$? The answer is given by demonstrating a distribution f which is not in $\mathcal{B}_0'$ but for which the sequence $\left\{U_j f\right\}$ does converge. This is done in the following examples.

<u>Example 1</u>: Let $f(t) = -\sum_{\nu=1}^{\infty} \delta^{(1)}(t - \nu)$. If ϕ is in $\mathcal{D}$, then $\left\langle f, \phi\right\rangle = \sum_{\nu=1}^{\infty} \phi^{(1)}(\nu)$; and the sum is actually finite since ϕ has compact support. In fact, if the support of ϕ is contained in $\left\{t : t \leqq K\right\}$, then

$$\lim_{j\to\infty} \left\langle U_j f, \phi\right\rangle = \lim_{j\to\infty}\left\langle -\sum_{\nu=1}^{\infty} \delta^{(1)}(t - \nu), \phi\left(\frac{t}{j}\right)\right\rangle = \lim_{j\to\infty} \sum_{\nu=1}^{jK} \frac{1}{j} \phi^{(1)}\left(\frac{\nu}{j}\right)$$

$$= \int_0^K \phi^{(1)}(t)\, dt = \phi(K) - \phi(0) = -\phi(0) = -\left\langle\delta, \phi\right\rangle$$

Therefore, $\lim_{j\to\infty} U_j f = -\delta$.

To see that f is not in $\mathscr{B}'_0$, look at the function $\phi(t) = \frac{\sin 2\pi t}{t}$. If $\phi(0)$ is defined by $\phi(0) = \lim_{t\to 0} \phi(t)$, then ϕ is in $\mathscr{B}_0$ since it is infinitely differentiable and each derivative approaches zero like $\frac{1}{t}$ as $|t| \to \infty$. However, $\left\langle f, \phi \right\rangle$ is not defined in this case since

$$\left\langle -\sum_{\nu=1}^{\infty} \delta^{(1)}(t-\nu), \frac{\sin 2\pi t}{t} \right\rangle = \sum_{\nu=1}^{\infty} \left(\frac{\sin 2\pi t}{t}\right)^{(1)}(\nu) = \sum_{\nu=1}^{\infty} \frac{2\pi\nu \cos 2\pi\nu - \sin 2\pi\nu}{\nu^2}$$

$$= \sum_{\nu=1}^{\infty} \frac{2\pi \cos 2\pi\nu}{\nu} = 2\pi \sum_{\nu=1}^{\infty} \frac{1}{\nu}$$

and this series does not converge. Thus $f(t) = \sum_{\nu=1}^{\infty} \delta^{(1)}(t-\nu)$ is a distribution not in $\mathscr{B}'_0$ for which the sequence $\left\{U_j f\right\}$ converges.

<u>Example 2:</u> The one-dimensional distribution p.v. $\frac{1}{t}$ is defined by

$$\left\langle \text{p.v. } \frac{1}{t}, \phi(t) \right\rangle = \lim_{\epsilon\to 0} \left[\int_{-\infty}^{-\epsilon} \frac{\phi(t)}{t}\, dt + \int_{\epsilon}^{\infty} \frac{\phi(t)}{t}\, dt \right]$$

where ϵ is always positive. This distribution is not in $\mathscr{B}'_0$ (since it obviously cannot be extended to all of $\mathscr{B}$) but it is invariant under all transformations of the type U_a where a is a positive real number. To see that this condition is true, let ϕ be in $\mathscr{D}$ and look at

$$\left\langle \text{p.v. } \frac{1}{t}, \phi(t) \right\rangle - \left\langle U_a\left(\text{p.v. } \frac{1}{t}\right), \phi(t) \right\rangle = \lim_{\epsilon\to 0} \left[\int_{-\infty}^{-\epsilon} \frac{\phi(t)}{t}\, dt + \int_{\epsilon}^{\infty} \frac{\phi(t)}{t}\, dt - \int_{-\infty}^{-\epsilon} \frac{\phi\left(\frac{t}{a}\right)}{t}\, dt - \int_{\epsilon}^{\infty} \frac{\phi\left(\frac{t}{a}\right)}{t}\, dt \right]$$

$$= \lim_{\epsilon\to 0} \left[-\int_{-\epsilon}^{-\frac{\epsilon}{a}} \frac{\phi(t)}{t}\, dt - \int_{\frac{\epsilon}{a}}^{\epsilon} \frac{\phi(t)}{t}\, dt \right] = \lim_{\epsilon\to 0} \left[\int_{\frac{\epsilon}{a}}^{\epsilon} \frac{\phi(-t) - \phi(t)}{t}\, dt \right]$$

$$= \lim_{\epsilon\to 0} \left[\int_{\frac{\epsilon}{a}}^{\epsilon} 2\phi^{(1)}(\theta(t))\, dt \right] = 0 \qquad \left(|\theta(t)| \leqq |t|\right)$$

Thus $U_a\left(\text{p.v. } \frac{1}{t}\right) = \text{p.v. } \frac{1}{t}$ for every positive real number a. The next theorem characterizes all distributions which are limits in $\mathscr{D}'(R)$ of sequences $\left\{U_j f\right\}$ as $j \to \infty$.

Theorem 6: If f is a one-dimensional distribution and $\left\{U_j f\right\}$ converges in $\mathscr{D}'$ to a distribution h, then

$$h(t) = c_1 \text{ p.v. } \frac{1}{t} + c_2\, \delta(t)$$

where c_1 and c_2 are constants.

Proof: Since $U_j f \to h$ in $\mathscr{D}'$ as $j \to \infty$, it is easy to see that $U_a h = h$ for every positive real number a. Therefore, if $a \neq 0$,

$$h(at) = \frac{1}{a} h(t) \tag{11}$$

Differentiating equation (11) with respect to a and evaluating the result at the point $a = 1$, gives $th^{(1)}(t) = -h(t)$. Therefore $(t\,h(t))^{(1)} = 0$, and by a familiar result on the differentiation of distributions (Horváth, ref. 12, p. 327) there is a constant c_1 such that

$$th(t) = c_1 \tag{12}$$

But for any constant c_1, the constant distribution $c_1(t)$ satisfies

$$c_1(t) = tc_1 \text{ p.v. } \frac{1}{t} \tag{13}$$

Thus, from equations (12) and (13) it follows that

$$t\left[h(t) - c_1 \text{ p.v. } \frac{1}{t}\right] = 0$$

which implies (Horváth, ref. 12, p. 352) that there is a constant c_2 such that

$$h(t) = c_1 \text{ p.v. } \frac{1}{t} + c_2\, \delta(t)$$

The proof is now complete.

For the Laplace transform of a distribution to be of any use, it must be analytic in some region of complex space. The next theorem will show that whenever f is such that the sequence $\left\{U_j T^{-p} f\right\}$ converges for p in a region of the complex plane, then the limit is the delta distribution multiplied by a function of p. Therefore, whenever the Laplace transform of f at p can be defined by using sequences of the form $\left\{U_j T^{-p} f\right\}$, the constant c_1 in theorem 6 must be zero.

<u>Theorem 7</u>: If there are two complex numbers p_1 and p_2 with $\mathrm{Re}\, p_1 \neq \mathrm{Re}\, p_2$ such that $\left\{U_j T^{-p_1} f\right\}$ and $\left\{U_j T^{-p_2} f\right\}$ both converge in $\mathcal{D}'_t$ as $j \to \infty$, then for every complex number p for which the sequence converges there is a constant c(p) such that

$$\lim_{j\to\infty} U_j T^{-p} f = c(p)\, \delta(t)$$

Proof: It may be assumed without loss of generality that $p_1 = 0$ and that $p_2 = p$ has a real part greater than zero. Let ϕ be a function in $\mathcal{D}$ whose support is contained in $(0,\infty)$, and for every positive integer j let $\phi_j(t) = e^{-pjt}\, \phi(t)$. Clearly, the sequence $\left\{\phi_j(t)\right\}$ converges to zero in $\mathcal{D}$ as $j \to \infty$.

By theorem 6 it is known that $\left\{U_j T^{-p} f\right\}$ converges to $c_1(p)\mathrm{p.v.}\, \frac{1}{t} + c_2(p)\, \delta(t)$; and since ϕ does not have support at the origin, $\left\langle \delta, \phi \right\rangle = 0$. Therefore,

$$\lim_{j\to\infty} \left\langle U_j T^{-p} f,\ \phi \right\rangle = \left\langle c_1(p)\mathrm{p.v.}\, \frac{1}{t},\ \phi(t) \right\rangle$$

But

$$\lim_{j\to\infty} \left\langle U_j T^{-p} f,\ \phi \right\rangle = \lim_{j\to\infty} \left\langle U_j\ f(t),\ e^{-pjt}\ \phi(t) \right\rangle = \lim_{j\to\infty} \left\langle U_j f,\ \phi_j \right\rangle = 0$$

by lemma 2 in the appendix since $\phi_j \to 0$ in $\mathcal{D}$ and $\left\{U_j f\right\}$ converges in $\mathcal{D}'$. Furthermore, the support of $\mathrm{p.v.}\, \frac{1}{t}$ is the whole real line, and the only way $\left\langle c_1(p)\mathrm{p.v.}\, \frac{1}{t},\ \phi(t) \right\rangle$

can equal zero for every ϕ with support contained in $(0,\infty)$ is for $c_1(p)$ to be zero. Thus $\lim_{j\to\infty} U_j T^{-p} f = c_2(p)\ \delta(t)$.

Now, let $g(t) = T^{-p} f(t)$; let ρ be a function in $\mathscr{D}$ with support contained in $(-\infty,0)$; and for every positive integer j, let $\rho_j(t) = e^{pjt}\rho(t)$. Then $\left\{\rho_j\right\}$ converges in $\mathscr{D}$ to zero and lemma 2 (see appendix) may be used as before to get

$$\lim_{j\to\infty} \left\langle U_j f, \rho\right\rangle = \lim_{j\to\infty} \left\langle U_j T^p g, \rho\right\rangle = \lim_{j\to\infty} \left\langle U_j g, \rho_j\right\rangle = 0$$

But $\lim_{j\to\infty} U_j f = c_1(0)\text{p.v.}\ \frac{1}{t} + c_2(0)\ \delta(t)$ and $\left\langle \delta, \rho\right\rangle = 0$, so $\left\langle c_1(0)\text{p.v.}\ \frac{1}{t}, \rho(t)\right\rangle = 0$. As before, the only way this can happen for all ρ in $\mathscr{D}$ with support contained in $(-\infty,0)$ is for $c_1(0)$ to be zero. Therefore, $\lim_{j\to\infty} U_j f = c_2(0)\ \delta$. Thus for every complex number p where the sequence converges,

$$\lim_{j\to\infty} U_j T^{-p} f = c(p)\ \delta$$

Corollary 1: If f is a distribution and there exist real numbers α and β such that $\left\{U_j T^{-p} f\right\}$ converges in $\mathscr{D}'$ as long as $\alpha < \text{Re } p < \beta$, then for each such complex number p,

$$\lim_{j\to\infty} U_j T^{-p} f = c(p)\ \delta$$

Corollary 2: If $\left\{U_j f\right\}$ converges in $\mathscr{D}'$ to $c_1\ \text{p.v.}\ \frac{1}{t} + c_2\ \delta(t)$ where $c_1 \neq 0$, then the sequence $\left\{U_j T^{-p} f\right\}$ cannot converge in $\mathscr{D}'$ as long as $\text{Re } p \neq 0$.

The purpose of this report is to use sequences of the form $\left\{U_j T^{-p} f\right\}$ to define the Laplace transform of f at p. Therefore, it would be helpful if corollary 1 could be strengthened by showing that if there are two complex numbers p_1, p_2 such that

$\left\{U_j T^{-p_1} f\right\}$ and $\left\{U_j T^{-p_2} f\right\}$ both converge, then as long as $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$, $\left\{U_j T^{-p} f\right\}$ also converges. This statement will follow from the next theorem which shows that whenever $\left\{U_j f\right\}$ converges in $\mathcal{D}'$, then f is in $\mathcal{S}'$.

<u>Theorem 8</u>: If f is a distribution such that the sequence $\left\{U_j f\right\}$ converges in $\mathcal{D}'$ as $j \to \infty$, then $f \in \mathcal{S}'$.

Proof: By lemma 3, in the appendix there are constants K and r such that if the support of ϕ is in the interval $[-1, 1]$, then for every j

$$\left|\left\langle U_j f, \phi\right\rangle\right| \leqq K \max_{|i| \leqq r} \sup \left|\phi^{(i)}\right|$$

If ϕ is in $\mathcal{D}$ with support contained in the interval $[-k, k]$, then the support of $\phi(kt)$ is in $[-1, 1]$; thus,

$$\left|\left\langle f, \phi\right\rangle\right| = \left|\left\langle U_k f, \phi(kt)\right\rangle\right| \leqq K \max_{|i| \leqq r} \sup_t \left|\left[\phi(kt)\right]^{(i)}\right| \leqq K k^r \max_{|i| \leqq r} \sup \left|\phi^{(i)}\right| \tag{14}$$

Now, let $\left\{\gamma_k\right\}$ be the partition of unity defined for R in the appendix and let θ be a function in $\mathcal{S}$. Then the function $\gamma_k \theta$ has support contained in the set $\left\{t : k - 1 \leqq |t| \leqq k + 1\right\}$; thus, by the properties of γ_k and inequality (14) it can be seen that

$$\left|\left\langle f, \gamma_k \theta\right\rangle\right| \leqq K(k+1)^r \max_{|i| \leqq r} \sup \left|\left[\gamma_k \theta\right]^{(i)}\right| \leqq KL(k+1)^r \max_{|i| \leqq r} \sup \left\{\left|\theta^{(i)}(t)\right| : k - 1 < t < k + 1\right\} \tag{15}$$

where $L = rr! \max_{|i| \leqq r} \sup \left|\gamma^{(i)}\right|$. Since θ is in $\mathcal{S}$, there is a constant K' such that

$$\max_{|i|\leq r} \left|\theta^{(i)}(t)\right| \leq \frac{K'}{\left(1 + |t|^2\right)^{r+2}} \tag{16}$$

for all t. So from expressions (15) and (16), it follows that

$$\left|\left\langle f, \gamma_k\theta\right\rangle\right| \leq KLK' \sup\left\{\frac{(k+1)^r}{\left(1+|t|^2\right)^{r+2}} : k - 1 < t < k + 1\right\} \leq KLK' \frac{(k+1)^r}{\left(1 + |k-1|^2\right)^{r+2}} \leq \frac{KLK'}{(k+1)^2}$$

as long as $k \geqq 3$. Therefore the series $\sum_{k=0}^{\infty} \left\langle f, \gamma_k\theta\right\rangle$ converges absolutely.

Since θ was an arbitrary function in $\mathcal{S}$, f may be extended to a functional on all of $\mathcal{S}$ by defining for any θ in $\mathcal{S}$

$$\left\langle f, \theta\right\rangle = \sum_{k=0}^{\infty} \left\langle f, \gamma_k\theta\right\rangle \tag{17}$$

If f were already in $\mathcal{S}'$, then expression (17) would be satisfied for every θ in $\mathcal{S}$; therefore, the definition is consistent. It is easy to see that expression (17) extends f in a linear and continuous fashion, so that f is in $\mathcal{S}'$, and the proof is complete.

<u>Corollary 3</u>: If there are two complex numbers p_1 and p_2 with $\mathrm{Re}\, p_1 < \mathrm{Re}\, p_2$ such that $\left\{U_j T^{-p_1} f\right\}$ and $\left\{U_j T^{-p_2} f\right\}$ both converge in $\mathcal{D}'$, then whenever $\mathrm{Re}\, p_1 < \mathrm{Re}\, p < \mathrm{Re}\, p_2$, $\left\{U_j T^{-p} f\right\}$ converges in $\mathcal{D}'$ to $\left\langle T^{-p}f, 1\right\rangle \delta$.

Proof: If $\left\{U_j T^{-p_1} f\right\}$ and $\left\{U_j T^{-p_2} f\right\}$ both converge in $\mathcal{D}'$, then by theorem 8, $T^{-p_1}f$ and $T^{-p_2}f$ are both in $\mathcal{S}'$. Also, by theorem 3, $T^{-p}f$ is in $\mathcal{B}_0'$ as long as $\mathrm{Re}\, p_1 < \mathrm{Re}\, p < \mathrm{Re}\, p_2$. Therefore, by theorem 5, $\lim_{j\to\infty} U_j T^{-p} f = \left\langle T^{-p}f, 1\right\rangle \delta$, and the corollary is proved.

THE LAPLACE TRANSFORM

Definition and Standard Results

In this section a new characterization of the Laplace transform for one-dimensional distributions is given. It will be used to prove the standard theorems concerning analyticity, uniqueness, and invertibility of the transform, and then to show that the new characterization is equivalent to Schwartz' definition of the Laplace transform for distributions which is given later in this section. However, the development given here is completely independent of Schwartz' treatment.

It will be said that a distribution f is Laplace transformable if there is an open interval (α,β) such that whenever p is a complex number with real part in (α,β), $T^{-p}f$ is a distribution in $\mathscr{B}'_0$. If (α,β) is the largest such open interval, then the set

$$\Omega = \left\{p : \text{ Re } p \in (\alpha,\beta)\right\}$$

will be called the domain of definition of the Laplace transform for f. The existence of the set Ω follows from theorem 3.

If f is a Laplace transformable distribution whose transform has domain of definition Ω, then for any $p \in \Omega$, the Laplace transform of f at p will be defined by

$$L\left[f\right](p) = \frac{1}{\phi(0)} \lim_{j\to\infty} \left\langle U_j T^{-p}f, \phi\right\rangle \tag{18}$$

where ϕ is a test function in $\mathscr{D}$ with $\phi(0) \neq 0$. Theorem 5 guarantees the existence of the limit in equation (18) and tells what it is. Thus, another characterization of the Laplace transform which is equivalent to equation (18) is

$$L\left[f\right](p) = \left\langle T^{-p}f, 1\right\rangle \tag{19}$$

By expression (19) it can be seen that $L\left[f\right]$ is a complex-valued function of the complex variable p with domain Ω. It also follows from expression (19) that the mapping L is linear. For, if f and g are distributions that are transformable at p and η and μ are complex numbers, then $\eta f + \mu g$ is Laplace transformable at p and

$$L\left[\eta f + \mu g\right] = \left\langle T^{-p}\left[\eta f + \mu g\right], 1\right\rangle = \eta\left\langle T^{-p}f, 1\right\rangle + \mu\left\langle T^{-p}g, 1\right\rangle = \eta L\left[f\right](p) + \mu L\left[g\right](p)$$

The next theorem shows that if f is Laplace transformable in Ω, then $L[f]$ is an analytic function of p in Ω.

Theorem 9: If f is a distribution that is Laplace transformable in Ω, then $L[f]$ is analytic in Ω and

$$\frac{d}{dp} L[f](p) = L[-t\, f(t)](p)$$

Proof: Suppose that $\Omega = \{p : \alpha < \mathrm{Re}\, p < \beta\}$; pick p_0 in Ω, and ϵ in $(0,1)$ such that $\epsilon < \min\{\mathrm{Re}\, p_0 - \alpha, \beta - \mathrm{Re}\, p_0\}$. If $\lambda(t) = e^{\epsilon t} + e^{-\epsilon t}$, then $1/\lambda$ is in $\mathcal{S} \subset \mathcal{B}_0$, and $\lambda T^{-p_0} f$ is in $\mathcal{B}_0'$. Also, as long as $|p - p_0| < \epsilon$, it follows that

$$\frac{L[f](p) - L[f](p_0)}{p - p_0} = \left\langle \frac{e^{-pt} - e^{-p_0 t}}{p - p_0} f(t), 1(t) \right\rangle = \left\langle \lambda(t) e^{-p_0 t} f(t), \frac{1}{\lambda(t)} \left[\frac{e^{-(p-p_0)t} - 1}{p - p_0}\right] \right\rangle$$

$$= \left\langle \lambda(t) e^{-p_0 t} f(t), \frac{-t}{\lambda(t)} + \frac{(p - p_0)t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{\left[-(p - p_0)t\right]^{j-2}}{j!} \right\rangle$$

Now, each derivative of $\dfrac{t^2}{\lambda(t)} \displaystyle\sum_{j=2}^{\infty} \frac{\left[-(p - p_0)t\right]^{j-2}}{j!}$ is bounded in absolute value by the corresponding derivative of $\dfrac{t^2}{\lambda(t)} e^{|(p-p_0)t|}$ and is therefore in $\mathcal{S}$. Thus, as $p \to p_0$, $\dfrac{1}{\lambda(t)} \left[\dfrac{e^{-(p-p_0)t} - 1}{p - p_0}\right]$ converges in $\mathcal{B}_0$ to $\dfrac{-t}{\lambda(t)}$ and

$$\frac{d}{dp} L\left[f\right]\left(p_0\right) = \lim_{p \to p_0} \frac{L\left[f\right](p) - L\left[f\right]\left(p_0\right)}{p - p_0} = \left\langle \lambda(t) T^{-p_0} f(t), \frac{-t}{\lambda(t)} \right\rangle$$

$$= \left\langle T^{-p_0}\left[-t\ f(t)\right], 1(t) \right\rangle = L\left[-t\ f(t)\right]\left(p_0\right)$$

This statement completes the proof of theorem 9.

Much of the usefulness of the Laplace transform is a result of the way it treats the convolution of two distributions. This important property of the transform is given by the next theorem.

Theorem 10: If f and g are Laplace transformable distributions such that the domains of their respective transforms have intersection Ω, then $f * g$ is Laplace transformable in Ω and for every p in Ω

$$L\left[f * g\right](p) = L\left[f\right](p)\ L\left[g\right](p)$$

Proof: For p in Ω, $T^{-p}f$ and $T^{-p}g$ are both in $\mathscr{B}_0'$; therefore, by theorem 4, $T^{-p}f * T^{-p}g = T^{-p}(f * g)$ is in $\mathscr{B}_0'$. Therefore $f * g$ is Laplace transformable at p; and from expression (19) and the definition of convolution, it can be seen that

$$L\left[f * g\right](p) = \left\langle T^{-p}(f * g), 1 \right\rangle = \left\langle T^{-p}f * T^{-p}g, 1 \right\rangle = \left\langle T^{-p} f(t) \otimes T^{-p} g(\tau), 1(t + \tau) \right\rangle$$

$$= \left\langle T^{-p} f(t) \otimes T^{-p} g(\tau), 1(t)\ 1(\tau) \right\rangle = \left\langle T^{-p}f, 1 \right\rangle \left\langle T^{-p}g, 1 \right\rangle = L\left[f\right](p)\ L\left[g\right](p)$$

which completes the proof.

No theory of the Laplace transform would be useful without inversion and uniqueness theorems. The next theorem will have these results as corollaries. In what follows, the real variable t and the real and imaginary parts of the complex variable p will serve at various times as independent variables. For this reason the particular independent variable for a space or an operation will be indicated by a subscript whenever this procedure will avoid ambiguity, for example, $\left\langle f(\tau), e^{-i\omega\tau} \right\rangle_\tau$ where $f(\tau)$ is in $\mathscr{B}'_{o,\tau}$ and ω is a parameter.

Theorem 11: If f is a distribution in $\mathscr{B}'_{o,t}$, then

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} \left\langle f(\tau), e^{-i\omega\tau} \right\rangle_\tau d\omega \tag{20}$$

where the limit is taken in $\mathscr{D}'_t$.

Proof: The integral in expression (20) is well defined since $\left\langle f(\tau), e^{-i\omega\tau} \right\rangle$ is a continuous function of ω. Let ϕ be in $\mathscr{D}_t$ and r be a positive real number. Then by standard theorems on the integration of distributions and test functions with respect to parameters, it follows that

$$\begin{aligned}
\left\langle \int_{-r}^{r} e^{i\omega t} \left\langle f(\tau), e^{-i\omega\tau} \right\rangle_\tau d\omega, \phi(t) \right\rangle_t &= \int_{-r}^{r} \left\langle e^{i\omega t} \left\langle f(\tau), e^{-i\omega\tau} \right\rangle_\tau, \phi(t) \right\rangle_t d\omega \\
&= \int_{-r}^{r} \left\langle f(\tau), e^{-i\omega\tau} \right\rangle_\tau \left\langle e^{i\omega t}, \phi(t) \right\rangle_t d\omega \\
&= \int_{-r}^{r} \left\langle f(\tau), \left\langle e^{i\omega(t-\tau)}, \phi(t) \right\rangle_t \right\rangle_\tau d\omega \\
&= \left\langle f(\tau), \int_{-r}^{r} \left\langle e^{i\omega(t-\tau)}, \phi(t) \right\rangle_t d\omega \right\rangle_\tau \\
&= \left\langle f(\tau), \int_{-r}^{r} e^{-i\omega\tau} \int_{-\infty}^{\infty} e^{i\omega t} \phi(t)\, dt\, d\omega \right\rangle_\tau \\
&= \left\langle f(\tau), \int_{-r}^{r} e^{i\xi\tau} \tilde{\phi}(\xi)\, d\xi \right\rangle_\tau
\end{aligned}$$

where $\xi = -\omega$ and $\tilde{\phi}(\xi)$ is the Fourier transform of $\phi(t)$. Clearly, as $r \to \infty$, $\int_{-r}^{r} e^{i\xi\tau} \tilde{\phi}(\xi)\, d\xi \to 2\pi\, \phi(\tau)$ uniformly with respect to τ, and similarly

$$\frac{d^k}{d\tau^k}\left[\int_{-r}^{r} e^{i\xi\tau}\,\tilde{\phi}(\xi)\,d\xi\right] = \int_{-r}^{r} (i\xi)^k e^{i\xi\tau}\,\tilde{\phi}(\xi)\,d\xi \to 2\pi\phi^{(k)}(\tau)$$

uniformly. Thus, the limit in $\mathscr{B}_\tau$ of $\int_{-r}^{r} e^{i\xi\tau}\,\tilde{\phi}(\xi)\,d\xi$ as $r \to \infty$ is $2\pi\,\phi(\tau)$, which means that

$$\left\langle \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} \left\langle f(\tau),\, e^{-i\omega\tau} \right\rangle_\tau d\omega,\ \phi(t) \right\rangle = \frac{1}{2\pi} \left\langle f(\tau),\ \lim_{r\to\infty} \int_{-r}^{r} \left\langle e^{i\omega(t-\tau)},\ \phi(t) \right\rangle_t d\omega \right\rangle_\tau$$

$$= \frac{1}{2\pi} \left\langle f(\tau),\ 2\pi\,\phi(\tau) \right\rangle_\tau = \left\langle f(t),\ \phi(t) \right\rangle_t$$

Thus, as distributions,

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} \left\langle f(\tau),\ e^{-i\omega\tau} \right\rangle_\tau d\omega$$

and the theorem is proved.

Corollary 4: If σ is a real number such that $e^{-\sigma t} f(t)$ is in $\mathscr{B}'_{o,t}$, then as distributions,

$$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} e^{pt} \left\langle e^{-p\tau} f(\tau),\ 1(\tau) \right\rangle_\tau dp$$

Proof: If $e^{-\sigma t} f(t)$ is in $\mathscr{B}'_{o,t}$, then as long as $\operatorname{Re} p = \sigma$, $e^{-pt} f(t)$ is in $\mathscr{B}'_{o,t}$, and

$$e^{-\sigma t} f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} \left\langle e^{-\sigma\tau} f(\tau),\ e^{-i\omega\tau} \right\rangle_\tau d\omega$$

Therefore,

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{\sigma t} e^{i\omega t} \left\langle e^{-\sigma\tau} f(\tau),\ e^{-i\omega\tau} \right\rangle_\tau d\omega = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\sigma-ir}^{\sigma+ir} e^{pt} \left\langle e^{-p\tau} f(\tau),\ 1(\tau) \right\rangle_\tau dp$$

which proves the corollary.

Corollary 5 (inversion theorem): If f is Laplace transformable in $\Omega = \left\{p : \alpha < \operatorname{Re} p < \beta\right\}$. Then, as long as $\alpha < \sigma < \beta$,

$$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} e^{pt}\, L\big[f\big](p)\, dp$$

where the limit is taken in $\mathscr{D}'_t$.

Corollary 6 (uniqueness theorem): If f and g are Laplace transformable distributions such that $L\big[f\big](p) = L\big[g\big](p)$ on some vertical line in the common domain of the transforms of f and g, then $f = g$ as distributions.

The next theorem gives sufficient conditions that an analytic function $F(p)$ be the Laplace transform of a distribution $f(t)$ and characterizes the distribution f.

Theorem 12: If $F(p)$ is analytic for p in

$$\Omega = \left\{\sigma + i\omega : \alpha < \sigma < \beta\right\}$$

and is bounded in Ω by a polynomial in ω $\big($or in $|p|\big)$, then $F(p) = L\big[f\big](p)$, where the distribution $f(t)$ is defined by

$$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} e^{pt}\, F(p)\, dp \tag{21}$$

for any fixed value of σ such that $\alpha < \sigma < \beta$.

Proof: The proof will be accomplished in four steps. It will be shown that (1) f is a distribution, (2) f is independent of the value of σ chosen in expression (21) as long as $\alpha < \sigma < \beta$, (3) $e^{-\sigma t} f(t)$ is in $\mathscr{B}'_{o,t}$ as long as $\alpha < \sigma < \beta$, and (4) $F(p) = L\big[f\big](p) = \left\langle T^{-p}f, 1\right\rangle$ for every p in Ω.

(1) To see that f is a distribution, let $\alpha < \sigma < \beta$ and let ϕ be in $\mathscr{D}_t$. Then

$$\left\langle \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} e^{pt}\, F(p)\, dp,\ \phi(t) \right\rangle_t = \left\langle \frac{1}{2\pi} \int_{-r}^{r} e^{\sigma t} e^{i\omega t} F(\sigma + i\omega)\, d\omega,\ \phi(t) \right\rangle_t$$

$$= \frac{1}{2\pi} \int_{-r}^{r} F(\sigma + i\omega)\mathscr{F}\Big[e^{-\sigma t}\, \phi(-t)\Big](\omega)\, d\omega \tag{22}$$

Now $e^{-\sigma t}\phi(-t)$ is in $\mathcal{D}_t$, so its Fourier transform is certainly in $\mathcal{S}_\omega$. Also, since $F(\sigma + i\omega)$ is a function bounded by a polynomial in ω, it is a regular distribution in $\mathcal{S}'_\omega$. Therefore, the limit as $r \to \infty$ of the last integral in expression (22) is well defined as the value of the regular distribution $F(\sigma + i\omega)$ at the testing function $\mathcal{F}\left[e^{-\sigma t}\,\phi(-t)\right]$, which means that

$$\left\langle f(t),\, \phi(t)\right\rangle = \left\langle \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\sigma-ir}^{\sigma+ir} e^{pt}\, F(p)\, dp,\, \phi(t)\right\rangle_t = \frac{1}{2\pi}\left\langle F(\sigma + i\omega),\, \mathcal{F}\left[e^{-\sigma t}\,\phi(-t)\right](\omega)\right\rangle_\omega \tag{23}$$

Clearly, if $\{\phi_k\}$ is a sequence that converges to zero in $\mathcal{D}_t$ as $k \to \infty$, then the sequence $\left\{\mathcal{F}\left[e^{-\sigma t}\,\phi_k(-t)\right]\right\}$ converges to zero in $\mathcal{S}_\omega$ as $k \to \infty$; so by expression (23), $\left\langle f,\, \phi_k\right\rangle \to 0$ also. Thus equation (21) defines f as a distribution.

(2) To see that f is independent of the choice of σ, choose σ_1, σ_2 such that $\alpha < \sigma_1 < \sigma_2 < \beta$; and for every positive real number r, let Γ_r be the closed path in Ω defined by the lines $\operatorname{Re} p = \sigma_1$, $\operatorname{Re} p = \sigma_2$, and $\operatorname{Im} p = \pm r$. Since $F(p)$ is analytic in Ω, Cauchy's theorem says that $\int_{\Gamma_r} e^{pt}\, F(p)\, dp = 0$. Therefore

$$\int_{\sigma_1-ir}^{\sigma_1+ir} e^{pt}\, F(p)\, dp - \int_{\sigma_2-ir}^{\sigma_2+ir} e^{pt}\, F(p)\, dp = \int_{\sigma_1-ir}^{\sigma_2-ir} e^{pt}\, F(p)\, dp + \int_{\sigma_2+ir}^{\sigma_1+ir} e^{pt}\, F(p)\, dp \tag{24}$$

But

$$\begin{aligned}\left\langle \int_{\sigma_1\pm ir}^{\sigma_2\pm ir} e^{pt}\, F(p)\, dp,\, \phi(t)\right\rangle &= \int_{\sigma_1}^{\sigma_2} \left\langle e^{(\sigma\pm ir)t}\, F(\sigma\pm ir),\, \phi(t)\right\rangle d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} F(\sigma \pm ir)\left\langle e^{(\sigma\pm ir)t},\, \phi(t)\right\rangle d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} F(\sigma \pm ir)\left\langle e^{\pm irt},\, e^{\sigma t}\,\phi(t)\right\rangle d\sigma \end{aligned} \tag{25}$$

Now $\left\langle e^{\pm irt}, e^{\sigma t}\, \phi(t)\right\rangle$ is a function in $\mathcal{S}_r$ for every value of σ, and the integral (25) is over a bounded interval; so as $r \to \infty$, the integral (25) approaches zero. Thus by expression (24) it can be seen that

$$\lim_{r\to\infty} \int_{\sigma_1 - ir}^{\sigma_1 + ir} e^{pt}\, F(p)\, dp = \lim_{r\to\infty} \int_{\sigma_2 - ir}^{\sigma_2 + ir} e^{pt}\, F(p)\, dp$$

as long as $\alpha < \sigma_1 < \sigma_2 < \beta$.

(3) In proving that $e^{-\sigma t}\, f(t)$ is in $\mathcal{B}'_{o,t}$ whenever $\alpha < \sigma < \beta$, part (2) of this proof, the fact that $F(\sigma + i\omega)$ is in $\mathcal{S}'_\omega$, and lemma 4 in the appendix will be used to get bounds on $\left\langle e^{-\sigma t}\, f(t), \phi(t)\right\rangle$ where ϕ is in $\mathcal{D}_t$. It can be seen that

$$\left|\left\langle e^{-\sigma t}\, f(t), \phi(t)\right\rangle\right| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + i\omega)\left\langle e^{i\omega t}, \phi(t)\right\rangle_t d\omega\right|$$

$$\leqq K_1 \sup_\omega \left|(1 + \omega^2)^{r_1} \frac{d^{r_1}}{d\omega^{r_1}} \left\langle e^{i\omega t}, \phi(t)\right\rangle_t\right|$$

$$= K_1 \sup_\omega \left|\sum_{k=0}^{r_1} \binom{r_1}{k} \omega^{2k} \left\langle e^{i\omega t}, (it)^{r_1}\, \phi(t)\right\rangle_t\right|$$

$$\leqq K_1 \sup_\omega \sum_{k=0}^{r_1} \binom{r_1}{k} |i\omega|^{2k} \left|\left\langle e^{i\omega t}, (it)^{r_1}\, \phi(t)\right\rangle_t\right|$$

$$\leqq K_1 r_1 r_1! \max_{|k| \leqq r_1} \sup_\omega \left|\left\langle e^{i\omega t}, \left[(it)^{r_1}\, \phi(t)\right]^{(2k)}\right\rangle_t\right|$$

$$\leqq K_1 r_1 r_1! \max_{|k| \leqq r_1} K_2 \max_{|j| \leqq r_2} \sup_t \left|(1 + t^2)^{r_2} \left[(it)^{r_1}\, \phi(t)\right]^{(2k+j)}\right| \tag{26}$$

where the last inequality follows from lemma 4. It is clear that the derivative of the product can be expanded by Leibnitz's rule and the various constants in expression (26) can be consolidated to get a positive number K and positive integer r which do not depend on ϕ such that

$$\left|\left\langle e^{-\sigma t} f(t),\ \phi(t)\right\rangle\right| \leqq K \max_{|j| \leqq r} \sup_{t} \left|(1 + t^2)^r\ \phi^{(j)}(t)\right|$$

This bound means that $e^{-\sigma t} f(t)$ is in $\mathscr{S}'_t$ for all σ such that $\alpha < \sigma < \beta$, and so by theorem 3, $e^{-\sigma t} f(t)$ is in $\mathscr{B}'_{o,t}$ for all such σ.

(4) Part (4) of this proof can be verified by using the first three parts and the uniqueness theorem for the inverse Fourier transform. However, it will be proved here by actually showing that $\left\langle T^{-p} f,\ 1\right\rangle = F(p)$. Let $p = \sigma + i\tau$ where $\alpha < \sigma < \beta$ and let ϕ be a function in $\mathscr{D}_t$ with $\phi(0) = 1$ and such that the support of ϕ is contained in $(-1, 1)$. Then by theorem 5,

$$\left\langle e^{-pt} f(t),\ 1(t)\right\rangle = \lim_{j\to\infty} \left\langle U_j e^{-pt} f(t),\ \phi(t)\right\rangle = \lim_{j\to\infty} \left\langle e^{-pt} f(t),\ \phi\left(\frac{t}{j}\right)\right\rangle$$

$$= \lim_{j\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma + i\omega) \left\langle e^{i(\omega-\tau)t},\ \phi\left(\frac{t}{j}\right)\right\rangle d\omega$$

Let $F = F_1 + F_2$ where the support of F_2 is contained in $\left\{\sigma + i\omega :\ |\omega - \tau| < 1\right\}$ and $F_2 = F$ in $\left\{\sigma + i\omega :\ |\omega - \tau| < \frac{1}{2}\right\}$. Also choose $k \geqq 2$ large enough to insure that

$G(\omega) = \dfrac{F_1(\sigma + i\omega)}{\left[i(\omega - \tau)\right]^k}$ is in L^1_ω, that is, an integrable function of ω. This can be done

since $F(\sigma + i\omega)$ is bounded by some polynomial in ω for p in Ω. Then it follows that

$$\lim_{j\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\sigma+i\omega)\left\langle e^{i(\omega-\tau)t},\ \phi\left(\frac{t}{j}\right)\right\rangle d\omega = \lim_{j\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\langle\left[i(\omega-\tau)\right]^k e^{i(\omega-\tau)t},\ \phi\left(\frac{t}{j}\right)\right\rangle_t d\omega$$

$$= \lim_{j\to\infty}\frac{(-1)^k}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\langle e^{i(\omega-\tau)t},\ \frac{1}{j^k}\,\phi^{(k)}\left(\frac{t}{j}\right)\right\rangle_t d\omega$$

$$= \lim_{j\to\infty}\frac{(-1)^k}{j^{k-1}}\int_{-j}^{j} e^{-i\tau t}\,\frac{\phi^{(k)}\left(\frac{t}{j}\right)}{j}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\,d\omega\,dt$$

$$= \lim_{j\to\infty}\frac{(-1)^k}{j^{k-1}}\int_{-j}^{j} e^{-i\tau t}\,\frac{\phi^{(k)}\left(\frac{t}{j}\right)}{j}\,\mathcal{F}^{-1}\left[G(\omega)\right](t)\,dt \qquad (27)$$

Since $G(\omega)$ is an L^1 function, its inverse Fourier transform is certainly bounded in absolute value, say by P. Therefore, the integrand in expression (27) is bounded in absolute value. Use of this bound and the choice of the test function ϕ gives the following:

$$\left|\int_{-j}^{j} e^{-i\tau t}\,\frac{\phi^{(k)}\left(\frac{t}{j}\right)}{j}\,\mathcal{F}^{-1}\left[G(\omega)\right](t)\,dt\right| \leqq \frac{P}{j}\int_{-j}^{j}\left|\phi^{(k)}\left(\frac{t}{j}\right)\right| dt \leqq \frac{P}{j}(2j)\ \sup\left|\phi^{(k)}\right| = 2P\ \sup\left|\phi^{(k)}\right|$$

Since $k \geqq 2$, it can be seen that the limit in expression (27) must be zero.

The term that has been neglected is

$$\lim_{j\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\sigma+i\omega)\left\langle e^{i(\omega-\tau)t},\ \phi\left(\frac{t}{j}\right)\right\rangle_t d\omega = \lim_{j\to\infty}\int_{-\infty}^{\infty} e^{-i\tau t}\,\phi\left(\frac{t}{j}\right)\mathcal{F}^{-1}\left[F_2(\sigma+i\omega)\right](t)\,dt \qquad (28)$$

Now $\mathcal{F}^{-1}\left[F_2(\sigma+i\omega)\right]$ is in $\mathcal{S}_t$, so as $j\to\infty$, $\left\{\phi\left(\frac{t}{j}\right)\mathcal{F}^{-1}\left[F_2(\sigma+i\omega)\right](t)\right\}$ converges in $\mathcal{S}_t$ to $\mathcal{F}^{-1}\left[F_2(\sigma+i\omega)\right]$. Also, $e^{-i\tau t}$ is a regular distribution in $\mathcal{S}_t'$, so the limit in expression (28) is

$$\int_{-\infty}^{\infty} e^{-i\tau t}\, \mathcal{F}^{-1}\Big[F_2(\sigma + i\omega)\Big](t)\, dt = F_2(\sigma + i\tau) = F(\sigma + i\tau) = F(p)$$

Thus it has been shown that

$$\left\langle T^{-p}f,\, 1\right\rangle = F(p)$$

as long as $\alpha < \operatorname{Re} p < \beta$, and the proof of theorem 12 is complete.

Comparison With Schwartz' Definition

The Laplace transform has been developed so far without any reference to the extension of the classical Laplace transform to distributions as defined by Schwartz. He defines the Laplace transform of a distribution f at $p = \sigma + i\omega$ by

$$L\Big[f\Big](p) = \mathcal{F}\Big[e^{-\sigma t}\, f(t)\Big](\omega)$$

In order to see that the development given here is equivalent to that of Schwartz, notice that by theorem 3 and the fact that $\mathcal{B}'_o \subset \mathcal{S}'$, a distribution f is Laplace transformable in the sense of this report if, and only if, $e^{-pt}\, f(t)$ is in $\mathcal{S}'_t$ for every p in Ω. Therefore, the transformable distributions and domains of the transform are the same for both definitions of the transform. Next, it can be seen that if there is an open interval (α, β) such that $e^{-\sigma t}\, f(t)$ in $\mathcal{B}'_{o,t}$ whenever σ is in (α, β), then the Fourier transform of $e^{-\sigma t}\, f(t)$ is an ordinary function of ω defined by

$$\mathcal{F}\Big[e^{-\sigma t}\, f(t)\Big](\omega) = \left\langle e^{-\sigma t}\, f(t),\, e^{-i\omega t}\right\rangle \tag{29}$$

The right-hand side of equation (29) makes sense as the application of a distribution in $\mathcal{B}'_o$ to a testing function in $\mathcal{B}$. To see that equation (29) is true, let ϕ be a function in $\mathcal{S}_\omega$. Then

$$\left\langle \mathcal{F}\Big[e^{-\sigma t}\, f(t)\Big](\omega),\, \phi(\omega)\right\rangle_\omega = \left\langle e^{-\sigma t}\, f(t),\, \tilde{\phi}(t)\right\rangle_t = \left\langle e^{-\sigma t}\, f(t),\, \int_{-\infty}^{\infty} e^{-i\omega t}\, \phi(\omega)\, d\omega\right\rangle_t$$

$$= \int_{-\infty}^{\infty} \left\langle e^{-\sigma t}\, f(t),\, e^{-i\omega t}\right\rangle_t \phi(\omega)\, d\omega = \left\langle \left\langle e^{-\sigma t}\, f(t),\, e^{-i\omega t}\right\rangle_t,\, \phi(\omega)\right\rangle_\omega$$

Thus, if f is Laplace transformable in Ω and $p = \sigma + i\omega$ is in Ω, Schwartz' definition of the transform gives

$$L\big[f\big](p) = \mathcal{F}\Big[e^{-\sigma t} f(t)\Big](\omega) = \Big\langle e^{-\sigma t} f(t),\ e^{-i\omega t}\Big\rangle_t = \Big\langle e^{-pt} f(t),\ 1(t)\Big\rangle_t = \Big\langle T^{-p} f,\ 1\Big\rangle$$

and the two definitions of the Laplace transform are equivalent.

Operation-Transform Formulas

Next, some of the standard operation-transform formulas for the distributional Laplace transform will be derived by using the characterization of the transform given in equation (18).

Let f be a Laplace transformable distribution whose transform has domain of definition $\Omega = \{p : \alpha < \operatorname{Re} p < \beta\}$. Then $f^{(1)}$ is also Laplace transformable in Ω. To compute the transform of $f^{(1)}$, let ϕ be a function in $\mathcal{S}$ such that $\phi(0) = 1$, $\phi'(0) \neq 0$, and let j be a positive integer. Then if p is in Ω,

$$\Big\langle U_j T^{-p} f^{(1)},\ \phi\Big\rangle = \Big\langle f^{(1)}(t),\ e^{-pt}\,\phi\Big(\frac{t}{j}\Big)\Big\rangle = \Big\langle f(t),\ pe^{-pt}\,\phi\Big(\frac{t}{j}\Big) - \frac{1}{j}\,e^{-pt}\,\phi^{(1)}\Big(\frac{t}{j}\Big)\Big\rangle$$

$$= p\Big\langle U_j T^{-p} f,\ \phi\Big\rangle - \frac{1}{j}\Big\langle U_j T^{-p} f,\ \phi^{(1)}\Big\rangle \tag{30}$$

As $j \to \infty$, the second term in the right-hand side of equation (30) converges to zero, and so it follows from equation (18) that

$$L\Big[f^{(1)}\Big](p) = \lim_{j\to\infty} p\Big\langle U_j T^{-p} f,\ \phi\Big\rangle = pL\big[f\big](p)$$

By an inductive argument it is easy to see that for every positive integer k,

$$L\Big[f^{(k)}\Big](p) = pL\Big[f^{(k-1)}\Big](p) = p^k L\big[f\big](p) \tag{31}$$

Another operational formula is furnished by theorem 9, which says that

$$L\Big[-t\,f(t)\Big](p) = \frac{d}{dp}\,L\big[f\big](p)$$

This formula can be extended by induction to get, for every positive integer k,

$$L\left[t^k f(t)\right](p) = (-1)^k \frac{d^k}{dp^k} L\left[f\right](p) \tag{32}$$

If f is Laplace transformable in Ω, then $f(t-\tau)$ is transformable in Ω for every real number τ, and

$$\left\langle U_j T^{-p} f(t-\tau),\ \phi(t)\right\rangle = \left\langle f(t-\tau),\ e^{-pt}\,\phi\left(\frac{t}{j}\right)\right\rangle = \left\langle f(t),\ e^{-p(t+\tau)}\,\phi\left(\frac{t+\tau}{j}\right)\right\rangle$$

$$= e^{-p\tau}\left\langle U_j T^{-p} f(t),\ \phi(t+\tau)\right\rangle$$

Now, $\phi(t+\tau)$ is in $\mathscr{D}$; and as long as $\phi(\tau) \neq 0$,

$$\lim_{j\to\infty} e^{-p\tau}\left\langle U_j T^{-p} f(t),\ \phi(t+\tau)\right\rangle = \frac{1}{\phi(\tau)}\, e^{-p\tau}\left\langle T^{-p} f,\ 1\right\rangle \left\langle \delta(t),\ \phi(t+\tau)\right\rangle$$

so

$$L\left[f(t-\tau)\right](p) = e^{-p\tau} L\left[f\right](p) \tag{33}$$

If q is a fixed complex number and f is Laplace transformable in Ω, then $e^{-qt} f(t)$ is Laplace transformable in $\Omega' = \left\{p :\ \alpha - \mathrm{Re}\, q < \mathrm{Re}\, p < \beta - \mathrm{Re}\, q\right\}$, and for p in Ω',

$$\left\langle U_j T^{-p}\left[e^{-qt} f(t)\right],\ \phi(t)\right\rangle = \left\langle U_j T^{-(p+q)} f,\ \phi\right\rangle$$

Therefore, as long as p is in Ω',

$$L\left[e^{-qt} f(t)\right](p) = L\left[f\right](p+q) \tag{34}$$

If k is a fixed positive integer and f is Laplace transformable in Ω, then $U_k f$ is Laplace transformable in $\Omega'' = \left\{p :\ k\alpha < \mathrm{Re}\, p < k\beta\right\}$. For $p \in \Omega''$ it can be seen that

$$\left\langle U_j T^{-p}\left[U_k f\right], \phi\right\rangle = \left\langle U_k f, e^{-pt}\,\phi\left(\frac{t}{j}\right)\right\rangle = \left\langle f(t), e^{-\frac{p}{k}t}\,\phi\left(\frac{t}{jk}\right)\right\rangle = \left\langle U_j T^{-\frac{p}{k}} f, \phi\left(\frac{t}{k}\right)\right\rangle$$

As $j \to \infty$, this sequence converges to $\left\langle T^{-\frac{p}{k}} f, 1\right\rangle \langle \delta, \phi\rangle$, which gives the formula

$$L\left[U_k f\right](p) = L\left[f\right]\left(\frac{p}{k}\right) \tag{35}$$

In order to demonstrate some of the theory developed so far, consider the distribution

$$f(t) = -\sum_{\nu=1}^{\infty} \delta^{(1)}(t - \nu)$$

Recall that in example 1 it was shown that f is not in $\mathscr{B}_0'$, but that $\left\{U_j f\right\}$ converges in $\mathscr{D}'$ to $-\delta$ as $j \to \infty$. Notice that if $\operatorname{Re} p > 0$, then $T^{-p}f$ is in $\mathscr{B}_0'$, so f has a Laplace transform defined in $\Omega = \left\{p : \operatorname{Re} p > 0\right\}$. Also notice that $f(t) = -\frac{d}{dt}\sum_{\nu=1}^{\infty} \delta(t - \nu)$; thus, by equation (31) if $g(t) = \sum_{\nu=1}^{\infty} \delta(t - \nu)$,

$$L\left[f\right](p) = -pL\left[g\right](p)$$

for every p in Ω.

If $\lambda(t)$ is a function in $\mathscr{B}$ such that $\lambda(t) = 0$ for $t < 0$ and $\lambda(t) = 1$ for $t \geqq \frac{1}{2}$, then by equation (19)

$$L\left[g\right](p) = \left\langle T^{-p}g, 1\right\rangle = \left\langle \sum_{\nu=1}^{\infty} \delta(t - \nu), e^{-pt}\,\lambda(t)\right\rangle = \sum_{\nu=1}^{\infty} e^{-p\nu} = e^{-p} \sum_{\nu=0}^{\infty} \left(e^{-p}\right)^{\nu} = \frac{1}{e^{p} - 1}$$

$$(p \in \Omega)$$

Therefore, for p in Ω,

$$L\left[f\right](p) = -pL\left[g\right](p) = \frac{p}{1 - e^p}$$

THE N-DIMENSIONAL LAPLACE TRANSFORM

Definitions

In this section the results proved in the preceding sections for distributions in $\mathscr{D}'(R)$ will be extended to $\mathscr{D}'(R^n)$. At the beginning of the third section, the linear transformations U_a and T^{-p} were defined, where a is a positive number and p is a complex number. If $a > 0$ in R^n $\left(a_i > 0,\ i = 1, 2, \ldots, n\right)$ define the linear transformation U_a on $\mathscr{D}'(R^n)$ by

$$\left\langle U_a f(t), \phi(t)\right\rangle = \left\langle a_1 a_2 \ldots a_n f\left(a_1 t_1, \ldots, a_n t_n\right), \phi\left(t_1, \ldots, t_n\right)\right\rangle$$
$$= \left\langle f\left(t_1, \ldots, t_n\right), \phi\left(\frac{t_1}{a_1}, \frac{t_2}{a_2}, \ldots, \frac{t_n}{a_n}\right)\right\rangle$$

The transformation T^{-p} is extended to $\mathscr{D}'(R^n)$ by the formula

$$\left\langle T^{-p} f(t), \phi(t)\right\rangle = \left\langle e^{-pt} f(t), \phi(t)\right\rangle = \left\langle e^{-\sum_i p_i t_i} f\left(t_1, \ldots, t_n\right), \phi\left(t_1, \ldots, t_n\right)\right\rangle$$

for each n-dimensional complex number $p = \left(p_1, \ldots, p_n\right)$.

Here, as in the third section, the major concern will be sequences of distributions of the form $\left\{U_j T^{-p} f\right\}$. However, in this section, j will represent a multi-index, $j = j_1, j_2, \ldots, j_n$ instead of a positive-integer-valued index. Let $j \to \infty$ mean that $j_1 \to \infty, j_2 \to \infty, \ldots, j_n \to \infty$, and for each i and k, $1 \leqq i \leqq n$ and $1 \leqq k \leqq n$, $j_i \to \infty$ independently of j_k. If $\left\{f_j\right\}$ is a "sequence" of distributions in $\mathscr{D}'(R^n)$ indexed by the multi-index j, then the statement

$$\lim_{j \to \infty} f_j = h$$

means that if ϕ is in $\mathscr{D}(R^n)$ and $\epsilon > 0$, then there is a positive integer N such that whenever $j_k \geqq N$ for every k, $1 \leqq k \leqq n$, then $\left| \langle f_j, \phi \rangle - \langle h, \phi \rangle \right| < \epsilon$.

The need for being very specific about what is meant by the limit of a sequence indexed by multi-indices will be demonstrated by the following example. Let the distribution h be defined by

$$h(t, \tau) = \frac{t^2 - \tau^2}{\left(t^2 + \tau^2\right)^2}$$

Then $h(t, \tau)$ is a rational function of t and τ with a removable singularity at the origin and can be considered a distribution in $\mathscr{D}'(R^2)$. It is easy to see that for every positive integer k,

$$h(t, \tau) = k^2 h(kt, k\tau)$$

Therefore, if $U_k h$ is defined by

$$U_k h(t, \tau) = k^2 h(kt, k\tau)$$

then $\lim\limits_{k \to \infty} U_k h = h$. However, it is not true that $\lim\limits_{j \to \infty} U_j h = h$, where j represents a multi-index of order 2. To verify this, let $j_k = (2k, k)$ for every positive integer k. Then

$$\lim_{k \to \infty} U_{j_k} h(t, \tau) = \lim_{k \to \infty} 2k^2 h(2kt, k\tau) = \lim_{k \to \infty} 2k^2 \left[\frac{4k^2 t^2 - k^2 \tau^2}{\left(4k^2 t^2 + k^2 \tau^2\right)^2} \right] = \frac{2\left(4t^2 - \tau^2\right)}{\left(4t^2 + \tau^2\right)^2} \neq h(t, \tau)$$

Thus, by the definition of the limit, h does not equal $\lim\limits_{j \to \infty} U_j h$.

Extensions of Results on U_j and T^{-p}

Since the results in this section are n-dimensional analogs of results already proved, only those for which the one-dimensional proofs do not generalize immediately will be proved here. In particular, theorem 5 may be generalized to n-dimensions without changing the statement or the proof significantly, so it will be accepted as an n-dimensional result without another proof.

The next theorem has a corollary which is the analog in n-dimensions of theorem 6 and its converse.

Theorem 13: If h is in $\mathcal{S}'(R^n)$, then $U_j h = h$ for every positive multi-index j if, and only if,

$$h(t) = \sum_{\nu=1}^{2^n} c_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1, . . ., n\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right) \tag{36}$$

for some constants c_ν, $1 \leqq \nu \leqq 2^n$.

Remark: In words, the theorem says that any distribution h in $\mathcal{S}'(R^n)$ which is invariant under each U_j is a linear combination of 2^n terms, each of which is the tensor product of n one-dimensional distributions of the form $\delta(t_i)$ or $\text{p.v.} \frac{1}{t_i}$. For example, if $n = 2$, then

$$h(t) = c_1 \text{ p.v.} \frac{1}{t_1} \otimes \text{p.v.} \frac{1}{t_2} + c_2 \text{ p.v.} \frac{1}{t_1} \otimes \delta(t_2) + c_3\, \delta(t_1) \otimes \text{p.v.} \frac{1}{t_2} + c_4\, \delta(t_1) \otimes \delta(t_2)$$

Proof of the theorem 13: The proof is by induction on n. If $n = 1$, then $U_j h = h$ for every positive multi-index j if, and only if, there is a distribution f such that $h = \lim_{j \to \infty} U_j f$. Therefore, the expansion (36) for k follows from theorem 6 in this case.
Let k be a positive integer and suppose that the theorem holds when $n = k - 1$. Let h be a distribution in $\mathcal{S}'(R^k)$ such that $U_j h = h$ for every positive multi-index j. If r_k denotes the multi-index $(0, 0, . . ., 0, r)$, where the r is in the kth position, then

$$U_{r_k} h = h \tag{37}$$

for every positive number r. Equation (37) may be differentiated with respect to r to get

$$\frac{d}{dr}\left(rh\left(t_1, \ldots, rt_k\right)\right) = \frac{d}{dr} h(t) = 0$$

or

$$h\left(t_1, \ldots, rt_k\right) + rt_k \frac{\partial h}{\partial t_k}\left(t_1, \ldots, rt_k\right) = 0 \tag{38}$$

Setting $r = 1$ in equation (38) gives

$$h(t) + t_k \frac{\partial h}{\partial t_k}(t) = \frac{\partial}{\partial t_k}\left(t_k\, h(t)\right) = 0 \tag{39}$$

Therefore, the distribution $t_k\, h(t)$ is independent of t_k, and by lemma 6 in the appendix, there is a distribution h_k in $\mathcal{D}'\left(R^{k-1}\right)$ such that

$$t_k\, h(t) = h_k\left(t_1, \ldots, t_{k-1}\right) \otimes 1\left(t_k\right)$$

Since $1\left(t_k\right) = t_k$ p.v. $\frac{1}{t_k}$, it follows that

$$t_k\left[h(t) - h_k\left(t_1, \ldots, t_{k-1}\right) \otimes \text{p.v. } \frac{1}{t_k}\right] = 0$$

By lemma 5 in the appendix, there must exist another distribution h_k' in $\mathcal{D}'\left(R^{k-1}\right)$ such that

$$h(t) - h_k\left(t_1, \ldots, t_{k-1}\right) \otimes \text{p.v. } \frac{1}{t_k} = h_k'\left(t_1, \ldots, t_{k-1}\right) \otimes \delta\left(t_k\right)$$

or

$$h(t) = h_k \otimes \text{p.v. } \frac{1}{t_k} + h_k' \otimes \delta\left(t_k\right)$$

Now, since $U_j h = h$ for every multi-index j, $h_k \otimes \text{p.v.} \frac{1}{t_k} + h_k' \otimes \delta(t_k)$ must also be invariant under each U_j. Therefore, if $\bar{j} = j_1, \ldots, j_{k-1}$, then

$$h_k \otimes \text{p.v.} \frac{1}{t_k} + h_k' \otimes \delta(t_k) = U_j\left(h_k \otimes \text{p.v.} \frac{1}{t_k}\right) + U_j\left(h_k' \otimes \delta(t_k)\right)$$

$$= U_{\bar{j}} h_k \otimes U_{j_k} \text{p.v.} \frac{1}{t_k} + U_{\bar{j}} h_k' \otimes U_{j_k} \delta(t_k)$$

$$= U_{\bar{j}} h_k \otimes \text{p.v.} \frac{1}{t_k} + U_{\bar{j}} h_k' \otimes \delta(t_k)$$

or

$$\text{p.v.} \frac{1}{t_k} \otimes \left(U_{\bar{j}} h_k - h_k\right) + \delta(t_k) \otimes \left(U_{\bar{j}} h_k' - h_k'\right) = 0$$

This can happen for every multi-index $\bar{j}$ of order $k - 1$ if, and only if, $U_{\bar{j}} h_k = h_k$ and $U_{\bar{j}} h_k' = h_k'$ for every $\bar{j}$. Since h_k and h_k' are both in $\mathscr{S}'(R^{k-1})$, the induction hypothesis says that there must be constants d_ν and d_ν', $1 \leqq \nu \leqq 2^{k-1}$, such that

$$h_k(t_1, \ldots, t_{k-1}) = \sum_{\nu=1}^{2^{k-1}} d_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1, \ldots, k-1\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right)$$

and

$$h_k'(t_1, \ldots, t_{k-1}) = \sum_{\nu=1}^{2^{k-1}} d_\nu' \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1, \ldots, k-1\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right)$$

Therefore

$$h(t) = h_k \otimes \text{p.v.}\,\frac{1}{t_k} + h'_k \otimes \delta(t_k) = \sum_{\nu=1}^{2^k} c_\nu \left(\underset{\substack{i \in I_\nu \\ I_\nu \subset \{1,\ldots,k\}}}{\otimes} \text{p.v.}\,\frac{1}{t_i} \right) \otimes \left(\underset{i \notin I_\nu}{\otimes} \delta(t_i) \right)$$

where the sequence $\{c_\nu\}_1^{2^k}$ is a rearrangement of the union of the two sequences

$\{d_\nu\}_1^{2^{k-1}}$ and $\{d'_\nu\}_1^{2^{k-1}}$.

Thus for every positive integer n, a representation of the form (36) holds for h in $\mathcal{D}'(R^n)$ whenever $U_j h = h$ for every positive multi-index j.

Conversely, if h has a representation of the form (36), then it is easy to see that $U_j h = h$ for every multi-index j. This completes the proof of theorem 13.

The observation that $h(t) = \lim_{j\to\infty} U_j f(t)$ for some distribution f in $\mathcal{D}'(R^n)$ if, and only if, $U_j h = h$ for every multi-index j gives an important corollary to theorem 13.

<u>Corollary 7</u>: If h is in $\mathcal{D}'(R^n)$, then $h = \lim_{j\to\infty} U_j f$ for some distribution f in $\mathcal{D}'(R^n)$ if, and only if, there exist constants c_ν, $1 \leqq \nu \leqq 2^n$ such that

$$h(t) = \sum_{\nu=1}^{2^n} c_\nu \left(\underset{\substack{i \in I_\nu \\ I_\nu \subset \{1,\ldots,n\}}}{\otimes} \text{p.v.}\,\frac{1}{t_i} \right) \otimes \left(\underset{i \notin I_\nu}{\otimes} \delta(t_i) \right)$$

Lemma 2 (see appendix) holds in $\mathcal{D}'(R^n)$ just as in $\mathcal{D}'(R)$ with virtually no change in the statement or proof. It will be used in the proof of the following theorem, which is an extension to n-dimensions of theorem 7.

<u>Theorem 14</u>: If f is in $\mathcal{D}'(R^n)$ and there are two complex numbers p_1, p_2 with $\text{Re}\, p_1 \neq \text{Re}\, p_2$ and a positive integer i,

$1 \leqq i \leqq n$ such that $\left\{U_j e^{-p_1 t_i} f(t)\right\}$ and $\left\{U_j e^{-p_2 t_i} f(t)\right\}$ both converge in $\mathscr{D}'(R^n)$ as the multi-index $j \to \infty$, then for every complex number q for which the sequence converges, there is a distribution $h(q)$ in $\mathscr{D}'\left(R^{n-1}\right)$ such that

$$\lim_{j\to\infty} U_j e^{-qt_i} f(t) = \delta\left(t_i\right) \otimes h(q) \tag{40}$$

Proof: It may be assumed, without loss of generality, that $p_1 = 0$ and that $p_2 = p$ has real part greater than zero. Define $h(0) = \lim_{j\to\infty} U_j f(t)$ and $h(p) = \lim_{j\to\infty} U_j e^{-pt_i} f(t)$. Let ϕ be a test function in $\mathscr{D}(R^n)$ with support $\phi \subset \left\{t : t_i > 0\right\}$. Then, if $j \to \infty$, clearly the sequence $\left\{e^{-pj_i t_i} \phi(t)\right\}$ converges to zero in $\mathscr{D}(R^n)$. Therefore, by lemma 2 (see appendix), it follows that

$$\lim_{j\to\infty} \left\langle U_j e^{-pt_i} f(t), \phi(t)\right\rangle = \lim_{j\to\infty} \left\langle U_j f(t), e^{-pj_i t_i}, \phi(t)\right\rangle = \left\langle h(0), 0\right\rangle = 0$$

But by corollary 7, there are constants c_ν, $1 \leqq \nu \leqq 2^n$ such that

$$h(p) = \sum_{\nu=1}^{2^n} c_\nu \left(\bigotimes_{\substack{i\in I_\nu \\ I_\nu \subset \{1,\ldots,n\}}} \text{p.v.} \frac{1}{t_i}\right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta\left(t_i\right)\right) \tag{41}$$

The only way a distribution of the form (41) can map every test function with support in $\left\{t : t_i > 0\right\}$ to zero is for the coefficient of every term in which the factor $\text{p.v.} \frac{1}{t_i}$ appears to be zero. Therefore, $h(p) = \delta\left(t_i\right) \otimes h'(p)$, where $h'(p)$ is in $\mathscr{D}'\left(R^{n-1}\right)$.

By using a similar argument, just as was done in the one-dimensional case, it can be shown that $h(0) = \delta\left(t_i\right) \otimes h'(0)$ for some distribution $h'(0)$ in $\mathscr{D}'\left(R^{n-1}\right)$. Thus, for any q at which the sequence $\left\{U_j e^{-qt_i} f(t)\right\}$ converges, its limit is of the form given by equation (40), and the theorem is proved.

<u>Corollary 8</u>: If $f \in \mathcal{S}'(R^n)$ is such that $\lim_{j\to\infty} U_j f = h(0)$ and for each i, $1 \leqq i \leqq n$, there is a complex p_i such that $\operatorname{Re} p_i \neq 0$ and $\lim_{j\to\infty} U_j e^{-p_i t_i} f(t) = h(p_i)$, then there is a constant c such that

$$h(0) = c\ \delta(t) = c\ \delta\left(t_1, \ldots, t_n\right)$$

Proof: By theorem 14 it can be seen that for each $i = 1, 2, \ldots, n$ there is a distribution $h_i(0)$ in $\mathcal{S}'\left(R^{n-1}\right)$ such that $h(0) = \delta\left(t_i\right) \otimes h_i(0)$. This can happen only if $h(0) = c\ \delta(t)$.

<u>Corollary 9</u>: Let Ω be an open set in C^n with the property that if p is in Ω, then the sequence $\left\{U_j T^{-p} f\right\}$ converges in $\mathcal{S}'(R^n)$ to a distribution $h(p)$ as $j \to \infty$. Then for every p in Ω there is a constant $c(p)$ such that

$$h(p) = c(p)\ \delta\left(t_1, t_2, \ldots, t_n\right)$$

Proof: Let p be in Ω and pick $\epsilon > 0$ such that the set $\left\{q : \left|q - p\right| < \epsilon\right\}$ is also in Ω. Let $g(t) = e^{-pt} f(t)$. Then $\lim_{j\to\infty} U_j\, g(t) = h(p)$ and $i = 1, 2, \ldots, n$,

$$\lim_{j\to\infty} U_j e^{-\frac{\epsilon}{2} t_i} g(t) = h\left({}_i p\right)$$

where, if $p = \left(p_1, p_2, \ldots, p_n\right)$, then ${}_i p = \left(p_1, p_2, \ldots, p_i + \frac{\epsilon}{2}, \ldots, p_n\right)$. Therefore, by corollary 8 $\lim_{j\to\infty} U_j\, g(t) = c(p)\ \delta(t)$, which completes the proof.

A generalization of theorem 8 to $\mathscr{D}'(R^n)$ does not change the statement of the theorem significantly; however, it is included here for completeness.

Theorem 15: If f is a distribution such that the sequence $\{U_j f\}$ converges in $\mathscr{D}'(R^n)$ as the multi-index $j \to \infty$, then f is in $\mathscr{S}'(R^n)$.

The proof of theorem 15 differs from that of theorem 8 only in details which are obvious. In particular, sets of the form $\{t : |t| \leqq k\}$ must be substituted for intervals $[-k, k]$, and the value of the constant L introduced in equation (15) must be adjusted. The statement and proof of lemma 3 (see appendix) do not change at all.

Corollary 10: If there are two n-dimensional complex numbers p_1 and p_2 with $\operatorname{Re} p_1 < \operatorname{Re} p_2$ such that $\{U_j T^{-p_1} f\}$ and $\{U_j T^{-p_2} f\}$ both converge in $\mathscr{D}'(R^n)$ as the multi-index $j \to \infty$, then whenever p is an n-dimensional complex number with $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$,

$$\lim_{j \to \infty} U_j T^{-p} f = \left\langle T^{-p} f, 1 \right\rangle \delta$$

The proof of corollary 10 follows from theorem 15, theorem 3, and theorem 5.

Extensions of Results on the Laplace Transform

The next topic to consider is the extension of the Laplace transform to distributions in $\mathscr{D}'(R^n)$. Since the definitions and theorems in the preceding section were based on the work done in previous sections, all of which has now been extended to n-dimensions, the extensions of the results on the Laplace transform are, for the most part, straightforward. The n-dimensional results will be stated without proof but the differences caused by going to $\mathscr{D}'(R^n)$ will be noted.

It will be said that a distribution f in $\mathscr{D}'(R^n)$ is Laplace transformable if there are two numbers a, b in R^n such that whenever p is an n-dimensional complex number with $a < \text{Re } p < b$, then $T^{-p}f$ is in $\mathscr{B}'_0(R^n)$. If (a, b) is the largest such n-dimensional open interval, then the set of n-dimensional complex numbers

$$\Omega = \left\{ p : \; a < \text{Re } p < b \right\} \quad .$$

will be called the domain of definition of the Laplace transform for f. The existence of the set Ω again follows from theorem 3.

The characterizations (18) and (19) of $L[f]$ in one dimension are also valid in n-dimensions; that is,

$$L[f](p) = \frac{1}{\phi(0)} \lim_{j \to \infty} \left\langle U_j T^{-p} f, \; \phi \right\rangle \tag{42}$$

where $p \in \Omega$ and ϕ is in $\mathscr{D}(R^n)$ with $\phi(0) \neq 0$, and

$$L[f](p) = \left\langle T^{-p} f, \; 1 \right\rangle \tag{43}$$

Formulas (42) and (43) are exactly the same as formulas (18) and (19) but are interpreted in n-dimensions. Clearly, $L[f]$ is a linear complex-valued function of the n-dimensional complex variable with domain Ω.

Theorem 9 on the analyticity of the transform may be extended to give

> <u>Theorem 16</u>: If $f \in \mathscr{D}'(R^n)$ is Laplace transformable in Ω, then $L[f]$ is analytic in Ω and
>
> $$\frac{\partial}{\partial p_i} L[f](p) = L\left[-t_i \; f(t) \right](p)$$

The proof of theorem 16 requires the use of Hartog's theorem (Bochner and Martin (ref. 13)) which says that a complex-valued function of n complex variables is analytic if it is analytic in each variable separately with all other variables held constant. The proof that $L[f]$ is analytic in each p_i separately is essentially the same as the proof of theorem 9.

The convolution theorem requires no change.

Theorem 17: If f and g are Laplace transformable distributions in $\mathscr{D}'(R^n)$ and the domains of their respective transforms have intersection Ω, then $f * g$ is Laplace transformable in Ω and for every p in Ω,

$$L[f * g](p) = L[f](p)\, L[g](p)$$

Theorem 18 (inversion theorem): If f is Laplace transformable in $\Omega = \left\{p : a < \operatorname{Re} p < b\right\}$, then for any fixed $\sigma \in R^n$ such that $a < \sigma < b$,

$$f(t) = \lim_{r \to \infty} \frac{1}{(2\pi i)^n} \int_{\sigma - ir}^{\sigma + ir} e^{pt}\, L[f](p)\, dp \tag{44}$$

where the limit is taken in $\mathscr{D}'_t(R^n)$ as $r \to \infty$ in R^n. The integral in equation (44) is taken over the subset of n-dimensional complex space defined by $\left\{p : \operatorname{Re} p_i = \sigma_i, \left|\operatorname{Im} p_i\right| < r_i, 1 \leqq i \leqq n\right\}$.

Theorem 19 (uniqueness theorem): If f and g are Laplace transformable distributions in $\mathscr{D}'(R^n)$ such that the domains of their transforms have intersection $\Omega = \left\{p : a < \operatorname{Re} p < b\right\}$, and there is a fixed $\sigma \in R^n$ with $a < \sigma < b$ such that whenever $\operatorname{Re} p = \sigma$, $L[f](p) = L[g](p)$; then $f = g$ as distributions.

Theorem 20: If $F(p)$ is analytic for p in $\Omega = \left\{p : a < \operatorname{Re} p < b\right\}$ and is bounded in Ω by a polynomial in $|\omega|$ $\left(\text{or in } |p|\right)$, then $F(p) = L[f](p)$ where the distribution f is defined as a limit in $\mathscr{D}'_t(R^n)$ by

$$f(t) = \lim_{r\to\infty} \frac{1}{(2\pi i)^n} \int_{\sigma-ir}^{\sigma+ir} e^{pt}\, F(p)\, dp \tag{45}$$

for any fixed $\sigma \in R^n$ such that $a < \sigma < b$.

Theorem 12, which is the one-dimensional analog of theorem 20, was proved in four steps, one of which required Cauchy's theorem. An n-dimensional analog of Cauchy's theorem can be found in Fuks (ref. 14).

The transform formulas developed in the fourth section also have n-dimensional analogs. For completeness, they are listed here. In the following formulas, k is a multi-index, t and τ are in R^n, p and q are n-dimensional complex numbers. Recall that $t^k = t_1^{k_1}, t_2^{k_2}, \ldots, t_n^{k_n}$, $\partial^k = \dfrac{\partial^{k_1+k_2+\ldots+k_3}}{\partial p_1^{k_1}, \partial p_2^{k_2}, \ldots, \partial p_n^{k_n}}$, and $\dfrac{p}{k} = \dfrac{p_1}{k_1}, \dfrac{p_2}{k_2}, \ldots, \dfrac{p_n}{k_n}$.

$$L\left[f^{(k)}\right](p) = p^k\, L\left[f\right](p) \tag{46}$$

$$L\left[t^k\, f(t)\right](p) = (-1)^{|k|}\, \partial^k\, L\left[f\right](p) \tag{47}$$

$$L\left[f(t-\tau)\right](p) = e^{-p\tau}\, L\left[f\right](p) \tag{48}$$

$$L\left[e^{-qt}\, f(t)\right](p) = L\left[f\right](p+q) \tag{49}$$

$$L\left[U_k f\right](p) = L\left[f\right]\left(\frac{p}{k}\right) \tag{50}$$

CONCLUDING REMARKS

A new characterization of the Laplace transform for Schwartz distributions is developed, by use of sequences of linear transformations on the space of distributions. The standard theorems on analyticity, uniqueness, and invertibility of the transform are proved by using the new characterization as the definition of the Laplace transform. It is shown that this sequential definition is equivalent to Schwartz' extension of the ordinary Laplace transform to distributions which he obtained from the Fourier transform.

Several theorems concerning dilatation transformations U_n and exponential shifts T^{-p} are proved. In particular, if f is an integrable distribution, then the sequence $U_j f$ converges to $\langle f, 1 \rangle \delta$ as j approaches ∞. Also, if f is a distribution such that $U_j f$ converges, then f must be a tempered distribution.

It is shown that a distribution h which is the limit as j approaches ∞ of a sequence $U_j f$ must be a linear combination of the delta distribution and the distribution p.v. $\frac{1}{t}$. Moreover, if $U_j T^{-p} f$ converges for two complex values of p having different real parts, then its limit is always a multiple of the delta distribution. This multiple turns out to be the Laplace transform of f at p.

All the results are extended to the n-dimensional case, but proofs are presented only for those situations that require methods different from their one-dimensional analogs.

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Hampton, Va., January 10, 1974.

APPENDIX

AUXILIARY RESULTS

This section contains several lemmas which are used in the first five sections, along with the construction of a partition of unity for R^n which satisfies certain special properties. In order to construct such a partition of unity, let $\lambda(t)$ be a function in $\mathcal{D}(R)$ that satisfies the following properties:

$$\lambda(t) \geqq 0 \quad \text{for every} \quad t \tag{A1}$$

$$\text{Support of} \quad \lambda(t) \subset \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{A2}$$

$$\lambda(t) = \lambda(-t) \quad \text{for every} \quad t \tag{A3}$$

$$\int_{-1/2}^{1/2} \lambda(t)\, dt = 1 \tag{A4}$$

An example of such a function is

$$\lambda(t) = \begin{cases} \dfrac{1}{A} \exp\left[\dfrac{1}{4t^2 - 1}\right] & \left(|t| < \dfrac{1}{2}\right) \\ 0 & \left(|t| \geqq \dfrac{1}{2}\right) \end{cases}$$

where

$$A = \int_{-1/2}^{1/2} \exp\left[\frac{1}{4t^2 - 1}\right] dt$$

Let the function $\rho(t)$ be defined by

$$\rho(t) = \int_{-\infty}^{t} \left[\lambda\left(\tau + \frac{1}{2}\right) - \lambda\left(\tau - \frac{1}{2}\right)\right] d\tau$$

Then $\rho \in \mathscr{D}(R)$, $\rho(0) = 1$, $\rho^{(j)}(0) = 0$ as long as $j \leqq 1$, $\rho(t) = \rho(-t)$ for all t, and support $\rho \subset \left[-1, 1\right]$. Also, if $t \in (0, 1)$

$$\rho(t) + \rho(t - 1) = \int_{-\infty}^{t} \left[\lambda\left(\tau + \frac{1}{2}\right) - \lambda\left(\tau - \frac{1}{2}\right)\right] d\tau + \int_{-\infty}^{t-1} \lambda\left(\tau + \frac{1}{2}\right) d\tau$$

$$= 1 - \int_{0}^{t} \lambda\left(\tau - \frac{1}{2}\right) d\tau + \int_{-1}^{t-1} \lambda\left(\tau + \frac{1}{2}\right) d\tau$$

$$= 1 - \int_{0}^{t} \lambda\left(\tau - \frac{1}{2}\right) d\tau + \int_{0}^{t} \lambda\left(\xi - \frac{1}{2}\right) d\xi = 1$$

where $\xi = \tau + 1$.

Now, for $t \in R^n$, define $\gamma_0(t) = \rho\left(\left|t\right|\right)$. Clearly, γ_0 is infinitely differentiable as long as $t \neq 0$. Define $\gamma_0^{(j)}(0) = 0$ for every multi-index j with $\left|j\right| > 0$ so that γ_0 is in $\mathscr{D}(R^n)$. For every positive integer k, define the function γ_k by

$$\gamma_k(t) = \rho\left(\left|t\right| - k\right)$$

Then the support of γ_k is contained in $\left\{t : k - 1 \leqq \left|t\right| \leqq k + 1\right\}$, and γ_k is in $\mathscr{D}(R^n)$ for every k. Also, if $k < \left|t\right| \leqq k + 1$, then

$$\sum_{\nu=0}^{\infty} \gamma_\nu(t) = \gamma_k(t) + \gamma_{k+1}(t) = \rho\left(\left|t\right| - k\right) + \rho\left(\left|t\right| - k - 1\right) = 1$$

since $\left|t\right| - k$ is in $(0, 1)$. Therefore $\left\{\gamma_k\right\}_{k=0}^{\infty}$ is a locally finite partition of unity which has the additional property that

$$\sup_t \left|\partial^j\left(\sum_{\nu \in I} \gamma_\nu\right)\right| \leqq \sup_t \left|\partial^j \gamma_0\right|$$

for any multi-index j and any subset I of nonnegative integers.

Next, a fact about complex numbers which is used in the proof of theorem 2 will be proved as a lemma.

<u>Lemma 1</u>: If $\left\{\eta_j\right\}_{j \in J}$ is a set of complex numbers with the property that there is a number P such that for every finite subset I of J

$$\left|\sum_{j \in I} \eta_j\right| \leqq P$$

then it is also true that

$$\sum_{j \in I} \left|\eta_j\right| \leqq 4P$$

for every finite subset I of J.

Proof: Suppose that there is a finite subset I' of J such that $\sum_{j \in I'} \left|\mathrm{Re}\ \eta_j\right| > 2P$. Then there must be a subset I'' of I' such that all the numbers $\mathrm{Re}\ \eta_j$ with $j \in I''$ have the same sign and

$$\left|\sum_{j \in I''} \mathrm{Re}\ \eta_j\right| = \sum_{j \in I''} \left|\mathrm{Re}\ \eta_j\right| > P$$

But by the hypothesis of the lemma

$$\left|\sum_{j \in I''} \mathrm{Re}\ \eta_j\right| \leqq \left|\sum_{j \in I''} \eta_j\right| \leqq P$$

which is a contradiction. Therefore, for every finite subset I of J, $\sum_{j \in I} \left|\mathrm{Re}\ \eta_j\right| \leqq 2P$, and similarly $\sum_{j \in I} \left|\mathrm{Im}\ \eta_j\right| \leqq 2P$. Thus

$$\sum_{j\in I}|\eta_j| \leqq \sum_{j\in I}\left|\text{Re } \eta_j\right| + \sum_{j\in I}\left|\text{Im } \eta_j\right| \leqq 4P$$

and the lemma is proved.

The next lemma concerns sequences of distributions and test functions. The result is known and in fact is a trivial consequence of the fact that the topology of $\mathscr{D}'$ is that of uniform convergence on bounded sets in $\mathscr{D}$. Since no topology for $\mathscr{D}'$ has been defined here, the lemma will be proved by modifying a standard proof of the completeness of $\mathscr{D}'$ (Gel´fand and Shilov (ref. 15)).

Lemma 2: If $\left\{f_k\right\}$ is a sequence that converges in $\mathscr{D}'$ to f and $\left\{\phi_k\right\}$ is a sequence that converges in $\mathscr{D}$ to ϕ, then the sequence of complex numbers $\left\{\left\langle f_k, \phi_k\right\rangle\right\}$ converges to $\left\langle f, \phi\right\rangle$.

Proof: It may be assumed without loss of generality that ϕ is the zero function; thus, it must be shown that $\left\{\left\langle f_k, \phi_k\right\rangle\right\}$ converges to zero. If the theorem is not true, then there must be a positive number c and a subsequence $\left(\text{denoted } \left\{\phi_\nu\right\} \text{ to save notation}\right)$ of $\left\{\phi_k\right\}$ such that for every ν

$$\left|\left\langle f_\nu, \phi_\nu\right\rangle\right| > c \qquad \text{(A5)}$$

It may also be assumed $\left(\text{if necessary by picking a subsequence of } \left\{\phi_\nu\right\}\right)$ that

$$\left|\partial^j \phi_\nu\right| < \frac{1}{4^\nu} \qquad (j = 1, 2, \ldots, \nu - 1) \qquad \text{(A6)}$$

Let $\psi_\nu = 2^\nu \phi_\nu$ for each ν and notice that $\left\{\psi_\nu\right\}$ converges to zero in $\mathscr{D}$ but $\left|\left\langle f_\nu, \psi_\nu\right\rangle\right| \to \infty$ as $\nu \to \infty$.

Now a subsequence $\left\{\psi_{k_\nu}\right\}$ of $\left\{\psi_\nu\right\}$ will be chosen as follows: Choose ψ_{k_1} such that $\left|\left\langle f_{k_1}, \psi_{k_1}\right\rangle\right| > 1$. This is possible since $\left|\left\langle f_\nu, \psi_\nu\right\rangle\right| \to \infty$. By assuming that ψ_{k_j} $(j = 1, 2, \ldots, \nu - 1)$ have been chosen, pick ψ_{k_ν} such that

$$\left|\left\langle f_{k_j}, \psi_{k_\nu}\right\rangle\right| < \frac{1}{2^{\nu-j}} \qquad (j = 1, 2, \ldots, \nu - 1) \tag{A7}$$

and

$$\left|\left\langle f_{k_\nu}, \psi_{k_\nu}\right\rangle\right| > \sum_{j=1}^{\nu-1} \left|\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| + \nu \tag{A8}$$

Equation (A7) can be satisfied since $\left\{\psi_\nu\right\} \to 0$ in $\mathscr{D}$ and equation (A8) can be satisfied since $\left|\left\langle f_\nu, \psi_\nu\right\rangle\right| \to \infty$.

Let $\psi = \sum_{j=1}^{\infty} \psi_{k_j}$. This series clearly converges in $\mathscr{D}$ by the way the ψ_k functions were defined. Then

$$\left\langle f_{k_\nu}, \psi\right\rangle = \sum_{j=1}^{\nu-1} \left\langle f_{k_\nu}, \psi_{k_j}\right\rangle + \left\langle f_{k_\nu}, \psi_{k_\nu}\right\rangle + \sum_{j=\nu+1}^{\infty} \left\langle f_{k_\nu}, \psi_{k_j}\right\rangle \tag{A9}$$

and by equation (A7) it may be seen that

$$\sum_{j=\nu+1}^{\infty} \left|\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| < \sum_{j=\nu+1}^{\infty} \frac{1}{2^{j-\nu}} = 1 \tag{A10}$$

From expression (A10), it follows that

$$\left|\left\langle f_{k_\nu}, \psi\right\rangle - \sum_{j=1}^{\nu-1}\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle - \sum_{j=\nu+1}^{\infty}\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| < \left|\left\langle f_{k_\nu}, \psi\right\rangle\right| + \sum_{j=1}^{\nu-1}\left|\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| + 1 \tag{A11}$$

and from expression (A8),

$$\left|\left\langle f_{k_\nu}, \psi\right\rangle - \sum_{j=1}^{\nu-1}\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle - \sum_{j=\nu+1}^{\infty}\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| = \left|\left\langle f_{k_\nu}, \psi_{k_\nu}\right\rangle\right| > \sum_{j=1}^{\nu-1}\left|\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| + \nu \tag{A12}$$

Combining expressions (A11) and (A12) gives

$$\left|\left\langle f_{k_\nu}, \psi\right\rangle\right| + \sum_{j=1}^{\nu-1}\left|\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| + 1 > \sum_{j=1}^{\nu-1}\left|\left\langle f_{k_\nu}, \psi_{k_j}\right\rangle\right| + \nu$$

or

$$\left|\left\langle f_{k_\nu}, \psi\right\rangle\right| > \nu - 1 \tag{A13}$$

This relation means that $\lim_{\nu\to\infty}\left|\left\langle f_\nu, \psi\right\rangle\right| = \infty$, which contradicts the hypothesis that $f = \lim_{\nu\to\infty} f_\nu$. Therefore, there can exist no subsequence $\left\{\phi_\nu\right\}$ of $\left\{\phi_k\right\}$ satisfying expression (A5). This statement completes the proof of lemma 2.

The next lemma is used in the proof of theorem 8.

<u>Lemma 3</u>: If $\left\{f_k\right\}$ is a sequence that converges in $\mathcal{D}'$ and C is a compact set in R, then there is a constant K and a positive integer r such that for every test function ϕ with support contained in C,

$$\left|\left\langle f_k, \phi\right\rangle\right| \leqq K \max_{|j|\leqq r} \sup\left|\phi^{(j)}\right| \tag{A14}$$

is satisfied for every k.

Proof: Let $\left\{f_k\right\}$ be a sequence that converges in $\mathscr{D}'$, and suppose there are no constants K and r such that expression (A14) is satisfied for every k. Then for each k there must be a test function ϕ_k whose support is contained in C such that

$$\left|\left\langle f_k, \phi_k \right\rangle\right| > k \max_{|j| \leqq k} \sup\left|\phi_k^{(j)}\right| \tag{A15}$$

For each k, let $\rho_k = \dfrac{\phi_k}{k \max\limits_{|j| \leqq k} \sup\left|\phi_k^{(j)}\right|}$. Then, if m is a positive integer,

$$\max_{|j| \leqq m} \sup\left|\rho_k^{(j)}\right| = \frac{\max\limits_{|j| \leqq m} \sup\left|\phi_k^{(j)}\right|}{k \max\limits_{|j| \leqq k} \sup\left|\phi_k^{(j)}\right|} \leqq \frac{1}{k}$$

as long as $k \geqq m$. Also the support of each ρ_k is in K; thus, $\left\{\rho_k\right\}$ converges to zero in $\mathscr{D}$. Therefore, by lemma 2, $\left\{\left\langle f_k, \rho_k \right\rangle\right\}$ converges to zero as $k \to \infty$. However, by expression (A15)

$$\left|\left\langle f_k, \rho_k \right\rangle\right| = \frac{\left|\left\langle f_k, \phi_k \right\rangle\right|}{k \max\limits_{|j| \leqq k} \sup\limits_{t}\left|\phi_k^{(j)}\right|} > 1$$

for every k; therefore, $\left\{\left\langle f_k, \rho_k \right\rangle\right\}$ cannot converge to zero, and a contradiction has been reached. Thus, no such sequence $\left\{\phi_k\right\}$ can exist, and the lemma is proved.

The next lemma establishes bounds for the Fourier transform of functions in $\mathscr{S}$. It is used in the proof of theorem 12.

Lemma 4: There exist a constant K and a positive integer r such that for every ϕ in $\mathcal{S}_t$,

$$\sup_{\omega}\left|\left\langle e^{i\omega t}, \phi(t)\right\rangle\right| \leqq K \max_{|j| \leqq r} \sup_{t}\left|\left(1+t^2\right)^r \phi^{(j)}(t)\right| \tag{A16}$$

Proof: The proof is by contradiction. If expression (A16) cannot be satisfied for all ϕ in $\mathcal{S}_t$ by any particular pair of constants K, r, then there must exist sequences $\left\{\phi_k\right\}$ in $\mathcal{S}_t$ and $\left\{\omega_k\right\}$ in R such that for every k,

$$\left|\left\langle e^{-i\omega_k t}, \phi_k(-t)\right\rangle\right| = \left|\left\langle e^{i\omega_k t}, \phi_k(t)\right\rangle\right| > k \max_{|j| \leqq k} \sup_{t}\left|\left(1+t^2\right)^k \phi_k^{(j)}(t)\right|$$

For each k, let $\psi_k(t) = \dfrac{\phi_k(-t)}{k \max\limits_{|j| \leqq k} \sup\limits_{t}\left|\left(1+t^2\right)^k \phi_k^{(j)}(t)\right|}$. Then $\psi_k \to 0$ in $\mathcal{S}_t$ as $k \to \infty$, and since the Fourier transform is a continuous mapping from $\mathcal{S}_t$ to $\mathcal{S}_\omega$, $\tilde{\psi}_k \to 0$ in $\mathcal{S}_\omega$ as $k \to \infty$. Therefore $\sup\left|\tilde{\psi}(\omega)\right| \to 0$ also. But $\left|\tilde{\psi}\left(\omega_k\right)\right| = \left|\left\langle e^{-i\omega_k t}, \psi_k(t)\right\rangle\right| > 1$ for every k, which is a contradiction.

Thus there can be no such sequences $\left\{\phi_k\right\}$ and $\left\{\omega_k\right\}$, and there must be a constant K and a positive integer r such that expression (A16) is satisfied for every ϕ in $\mathcal{S}_t$. This statement completes the proof of lemma 4.

In the last section, two lemmas which are standard results in $\mathcal{D}'(R^n)$ are required and are stated here for convenience. The proofs of these lemmas are straightforward and they will not be given here. (See ref. 12, p. 380.)

Lemma 5: If f is in $\mathcal{D}'(R^n)$, then $t_i f = 0$ if, and only if,

$$f = \delta\left(t_i\right) \otimes g$$

where g is in $\mathcal{D}'\left(R^{n-1}\right)$.

Lemma 6: The distribution f in $\mathcal{D}'(R^n)$ is independent of the variables $t_1, \ldots, t_k$ if, and only if,

$$f(t_1, \ldots, t_n) = 1(t_1, \ldots, t_k) \otimes g(t_{k+1}, \ldots, t_n)$$

where $1(t_1, \ldots, t_k)$ is the function which takes the constant value 1 on R^k and g is in $\mathcal{D}'(R^{n-k})$.

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